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Patrick Joly, Adrien Semin

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Abstract: In this research report, we analyze via the theory of matched asymptotics the propagation of a time harmonic acoustic wave in a junction of two thin slots. This allows us to propose improved Kirchoff conditions for the 1D limit model. These conditions are analyzed and validated numerically

Key-words: Matched asymptotics, thin slots, Kirchoff conditions

* Equipe-projet POems - mail : patrick.joly@inria.fr & adrien.semin@inria.fr

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex (France)
Téléphone : +33 1 39 63 55 11 — Télécopie : +33 1 39 63 53 30

propagation d'une onde acoustique dans une jonction de fentes minces

Résumé : Dans ce rapport de recherche, nous utilisons la théorie des développements asymptotiques raccprdés pour analyser la propagation d'ondes acoustiques à travers une jonction de deux fentes minces. Ceci nous permet de proposer des conditions de Kirchoff améliorées pour le problème limite 1D. Ces conditions sont analysées et validées numériquement.

Ce rapport de recherche contient également les démonstrations de [JS08], et la trame de ce rapport suit en grande partie la trame de l'article cité.

Mots-clés : Développements asymptotiques raccordés, fentes minces, conditions de Kirchoff

Introduction

One can consider time harmonic wave propagation and time domain wave propagation in thin domains that are junctions of thin slots whose thickness ε is small with respect to the wave length λ and converge, when ε tends to 0 to a 1-dimensional graph. Intuitively, one expects that the solution of the original model converges (in a sense we can find in the works of Rubinstein and Schatzman [RS01a], [RS01b] or Kuchment [Kuc02]) to a 1D function defined on the limit graph. The homogeneous Neumann boundary condition is considered (the "good" boundary conditions from a physical point of view). The limit model is known for a long time: the limit solution satisfies the 1D time harmonic wave equation (namely the Helmholtz equation), or the 1D time domain wave equation, and the so called Kirchoff conditions (in electricity) at each node of the graph: the solution is continuous at the node of the graph and the sum of fluxes at this node vanishes. Note however that the rigorous justification of such a result in quite recent: see the works of Rubinstein and Schatzman [RS01a], [RS01b] or Kuchment [Kuc02]. In these papers, eigenvalue problems are considered (but the results are quite similar to the ones that we shall investigate) : the convergence to the limit problem is established (without error estimate).

This research report intends to covert the simpliest case : the study of aymptotic models for time harmonic wave propagation and time domain wave propagation in thin domains that are junctions of thin slots whose thickness ε is small with respect to the wave length λ and converge, when ε tends to 0 to a 1-dimensional graph. Intuitively, one expects that the solution of the original model converges (in a sense that will be made more precise later in this report) to a 1D function defined on the limit graph. The homogeneous Neuman boundary condition is considered. A natural question is to look for more accurate approximate models, i.e. models that would permit to identify not only the limit solution but also its first order (or higher order) expansion with respect to ε. As we shall see, this can be reduced to constructing improved Kirchoff conditions at the nodes ot the limit graph. The present work is a first contribution in this direction in the simplest possible case where we consider scalar wave propagation in a homogeneous medium composed by the junction of two thin 2D slots of the same thickness: in particular, the limit graph has only two branches and one node, From the technical point of view, the interest is that, even in this very simple case, the analysis is not so trivial and going beyond the limit problem requires some multi-scale asymptotic analysis in order to capture the non 1D phenomena that take place in the neighborhood of the junction. In this paper, in the spirit of the work of [JT06] for the junction between a thin slot and a half-space, we shall use the method of matched asymptotics (see also [Il′92], [VD64] for more general references) which is an interesting alternative to multiscale expansions (see [MNP00] and [TVD06] for the connection between the two approaches), and we extend the results we got for the time harmonic wave equation to the time domain wave equation.

The outline of the report is as follow :

- in the section 1, we present the problem we study in this report, and we claim our main results,
- in the section 2, we express the exact solution of our problem as a development of functions with respect with the variable $\varepsilon\omega$, and we write the problems satisfied by the various terms of our expansions,
- in the section 3, we prove rigorously that the terms of our expansions are well-defined, i.e. they satisfy canonical problems that we know, with various techniques from the PDE's, they have a unique solution; and we prove that the exact solution of the initial problem differs from the truncated sum of the terms of our expansion with a constant which get smaller and smaller with the increasing of the number of terms we take,
- in the section 4, we show that on the most interesting part of the domain we consider, the solution can be computed by a simplier model - we explain this model and which error we have by taking this model instead of taking the full problem,
- the sections 5 and 6 show many theorical and numerical results that illustrate the previous sections.

We put in the appendix A to C some technical results that are used many times along this report and that may (and will) be used for future papers.

1 Model problem and main results

1.1 Geometry of the domain

In this section we introduce the geometry and the equations of our problem. We consider a domain made of the junction of two slots (see figure 1.1). More precicely, we consider the union of two thin rectangles with respective lengths L_- and L_+ and thickness ε, the small parameter in the analysis, with a junction zone. A geometrical characteristic of this domain is the angle 2α between the two thin slots. For the analysis, we consider in fact a family of such thin domains denoted $\Omega_\alpha^\varepsilon$ with varying ε. We make the choice (this has an influence on the asymptotic analysis) that one part of the boundary of $\Omega_\alpha^\varepsilon$ remains fixed, namely the two segments S^- and S^+ that intersect at the reentrant corner of the junction zone. Analytically, we have:

$$\Omega_\alpha^\varepsilon = \Omega_-^\varepsilon \cup \Omega_+^\varepsilon \cup J_\alpha^\varepsilon \tag{1.1}$$

with $J_\alpha^\varepsilon = \varepsilon\, \widehat{J}_\alpha$ where $\widehat{J}_\alpha$ is the normalized junction presented in figure 1.2 and

$$\left\{ \begin{array}{l} \Omega_\pm^\varepsilon = \left\{ \mathbf{x} = (x,y) \in \mathbb{R}^2 \;/\; 0 < \pm\, \mathbf{x}\cdot\mathbf{t}^\pm < L_\pm,\ -\varepsilon < \mathbf{x}\cdot\mathbf{n}^\pm < 0 \right\} \\[2mm] \mathbf{t}^- = (1,0)^t, \quad \mathbf{n}^- = (0,1)^t, \\[2mm] \mathbf{t}^+ = \big(\cos(2\alpha), \sin(2\alpha)\big)^t, \quad \mathbf{n}^+ = \big(-\sin(2\alpha), \cos(2\alpha)\big)^t, \end{array} \right. \tag{1.2}$$

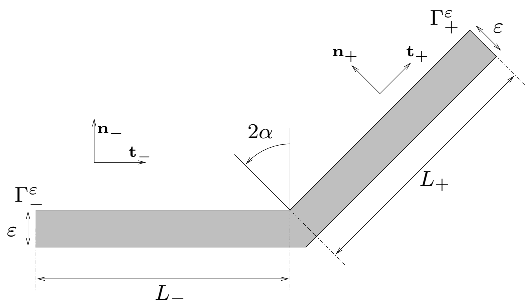

Figure 1.1: Configuration of the domain $\Omega_\alpha^\varepsilon$

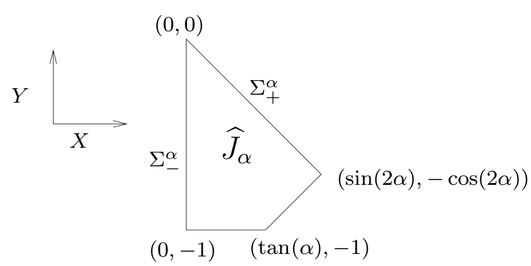

Figure 1.2: The normalized junction $\widehat{J}_\alpha$

Remark 1.1. It is possible to take some other shapes for the junction, we will see later where this has an influence.

The problem we consider is : find $u^\varepsilon \in \mathrm{H}^1(\Omega_\alpha^\varepsilon; \mathbb{C})$ such as

$$\left\{ \begin{array}{llll} -\Delta u^\varepsilon - \omega^2 u^\varepsilon = 0, & \text{in } \Omega_\alpha^\varepsilon & \dfrac{\partial u^\varepsilon}{\partial n} = f, & \text{on } \Gamma_-^\varepsilon, \\[3mm] \dfrac{\partial u^\varepsilon}{\partial n} - \imath\,\omega u^\varepsilon = 0, & \text{on } \Gamma_+^\varepsilon & \dfrac{\partial u^\varepsilon}{\partial n} = 0, & \text{on } \partial\Omega_\alpha^\varepsilon \setminus (\Gamma_-^\varepsilon \cup \Gamma_+^\varepsilon). \end{array} \right. \tag{1.3}$$

where the source term f is a given constant (i.e. a constant function along Γ^ε_-) and n denotes the outward unit vector along $\partial\Omega^\varepsilon_\alpha$.

Remark 1.2. The third line of (1.3) describes that the unknown function u^ε verifies the Sommerfield radiation condition. The reader can see [KG89] for more details. Here we use an approximate order 1 condition.

1.2 The 1D limit problem

When ε tends to 0, the domain Ω^ε "degenerates" into a "1D domain" namely the union of the two segments S_- and S^+. Intuitively, one expects that the solution u^ε "converges" to a "1D function", namely a function of the arclength s along $S_- \cup S_+$, solution of a "1D problem". It remains to give a more precise mathematical meaning to such a statement. To describe the "limit problem" inside the slots, we will use local normalized (tangential and normal) coordinates $(s, \widehat{\nu})$, that express that Ω^ε_+ and Ω^ε_- are isomorphic to the rectangles

$$\widehat{\Omega}_+ =]0, L_+[\times] - 1, 0[\,, \quad \widehat{\Omega}_- =] - L_-, 0[\times] - 1, 0[\,,$$

through the maps

$$\mathbf{x} = (x, y) \mapsto (s, \widehat{\nu}) = (\mathbf{x} \cdot \mathbf{t}^\pm, \mathbf{x} \cdot \mathbf{n}^\pm/\varepsilon) \quad \text{from } \Omega^\varepsilon_\pm \text{ into } \widehat{\Omega}_\pm.$$

Let us introduce $u^0(s) :] - L_-, L_+[\to \mathbb{C}$ be the solution of the 1D Hemholtz equation (our 1D limit problem):

$$\left\{ \begin{array}{lll} \dfrac{\partial^2 u^0}{\partial s^2} + \omega^2 u^0 = 0, & \text{in }] - L_-, L_+[\,, & (i) \\[2ex] -\dfrac{\partial u^0}{\partial s}(-L_-) = f, & \left[\dfrac{\partial u^0}{\partial s} - \imath\omega u^0\right](L_+) = 0, & (ii) \end{array} \right. \tag{1.4}$$

from which we define the 2D functions $\widehat{u}^0_\pm : \widehat{\Omega}_\pm \to \mathbb{C}$ such that

$$\widehat{u}^0_\pm(s, \widehat{\nu}) = u^0(s) \quad \text{in } \widehat{\Omega}_\pm \tag{1.5}$$

Next, we introduce the "rescaled" exact solutions in $\widehat{\Omega}_\pm$

$$\widehat{u}^\varepsilon_\pm(s, \widehat{\nu}) = u^\varepsilon(x, y) \quad \text{for } (s, \widehat{\nu}) = (\mathbf{x} \cdot \mathbf{t}^\pm, \mathbf{x} \cdot \mathbf{n}^\pm/\varepsilon) \tag{1.6}$$

as well as in the normalized junction

$$\widehat{U}^\varepsilon(\widehat{\mathbf{x}}) = u^\varepsilon(\varepsilon\,\widehat{\mathbf{x}}) \quad \text{in } \widehat{J}_\alpha. \tag{1.7}$$

The precise meaning of the convergence of u^ε to u^0 is given in the following theorem :

Theorem 1.3. *The functions $\widehat{u}_{\pm}^{\varepsilon}$ converge to the "1D functions" $\widehat{u}_{\pm}^{0}$ in $\mathrm{H}^1(\widehat{\Omega}_{\pm})$ and the function $\widehat{U}^{\varepsilon}$ converges to the constant $\widehat{u}^0(0)$ in $\mathrm{H}^1(\widehat{J}_\alpha)$. Moreover, there exists a positive constant C, independent of ε such that:*

$$\sum_{\pm} \|\widehat{u}_{\pm}^{\varepsilon} - \widehat{u}_{\pm}^{0}\|_{\mathrm{H}^1(\widehat{\Omega}_{\pm})} + \|\widehat{U}^{\varepsilon} - \widehat{u}^0(0)\|_{\mathrm{H}^1(\widehat{J}_\alpha)} \leqslant C\,\varepsilon. \tag{1.8}$$

Proof. We first prove that we have some stability result (see the appendix B). By looking more carefully the proof of the proposition B.1, we can see that the derivate of $\widehat{u}_{\pm}^{\varepsilon}$ over $\widehat{\nu}$ and the gradient of $\widehat{U}^{\varepsilon}$ converges to 0, as ε converges to 0, and the fact that the limit solution belongs to $\mathrm{H}^1(\widehat{\Omega}_- \cup \widehat{J}_\alpha \cup \widehat{\Omega}_+)$ gives that $\widehat{U}^0 = \widehat{u}^0(0)$. To conclude, we multiply the first equation of (1.3) by the limit function $\widehat{u}^0$, and by applying the Green formula, we can see that the derivate of the limit function over s is continuous at $s = 0$, then the function u^0 satisfies the problem (1.4).
The proof of the error estimate is much less trivial and will be in fact a consequence of the asymptotic analysis developed in the present work. Let us remark that this estimate can not be deduced from the results of [Kuc02] or [RS01a]. ■

A first immediate (and a priori surprising) observation is that the limit solution u^0 does not see the geometry of the junction: in particular, it is independent of the angle α ! Physically it means that, for the limit problem, the incident wave emitted at $s = -L^-$ by the source term f does not produce any reflection when it reaches the junction at $s = 0$: it is completely transmitted. For $\varepsilon > 0$, it is clear that there exists a reflected wave whose amplitude is expected to be an increasing function of α. This raises the following question: does there exist an "improved 1D model" whose solution would provide a better approximation to the exact solution than u^0, and would in particular permit us to observe a reflected wave ? The answer to this question (at least one possible answer) is the object of the following subsection.

To better understand how the 1D problem will be modified, one has to interpret the limit solution u^0 as the solution of a transmission problem between two 1D Helmholtz equations in the segments $]-L^-, 0[$ and $]0, L^+[$ with the help of the transmission conditions:

$$\left\{ \begin{aligned} & \left[u^0\right] := u^0(0^+) - u^0(0^-) = 0, \\ & \left[\frac{\partial u^0}{\partial s}\right] := \frac{\partial u^0}{\partial s}(0^+) - \frac{\partial u^0}{\partial s}(0^-) = 0. \end{aligned} \right. \tag{1.9}$$

These are nothing but the particular version of the well-known "Kirchoff conditions" at a node where only two branches of a graph is connected. Thus, our question could be rephrased as follows : does there exist "improved Kirchoff conditions" that would lead to a better approximation of the true solution than u^0 ?

1.3 An improved 1D approximate model

To describe our improved model, we need to introduce some additional notation. The construction of the improved solution will use the solution of an auxiliary (frequency independent) boundary value problem posed in the normalized junction $\widehat{J}_\alpha$, namely the problem which consists in finding $\mathcal{W}_\alpha \in \mathrm{H}^1(\widehat{J}_\alpha)$, with mean-value 0

$$\int_{\widehat{J}_\alpha} \mathcal{W}_\alpha = 0, \tag{1.10}$$

that solves the boundary value problem (note that $\mathcal{W}_\alpha$ depends on α through $\widehat{J}_\alpha$)

$$\begin{cases} \Delta \mathcal{W}_\alpha = 0, & \text{in } \widehat{J}_\alpha, \\ \dfrac{\partial \mathcal{W}_\alpha}{\partial n} + T_\pm \mathcal{W}_\alpha = \pm 1, & \text{on } \Sigma_\pm^\alpha, \\ \dfrac{\partial \mathcal{W}_\alpha}{\partial n} = 0, & \text{on } \partial \widehat{J}_\alpha \setminus \left(\Sigma_+^\alpha \cup \Sigma_-^\alpha\right) \end{cases} \tag{1.11}$$

where, once $\Sigma_\pm^\alpha$ has been identified to the segment $]-1,0[$, $T_\pm$ is nothing but the nonlocal operator defined as

$$\left| \begin{array}{llll} T: & \mathrm{H}^{\frac{1}{2}}(\,]-1,0[\,) & \to & \mathrm{H}^{-\frac{1}{2}}(\,]-1,0[\,) \\ & \varphi = \displaystyle\sum_{p=0}^{\infty} \varphi_p \, w_p & \mapsto & T\varphi = \displaystyle\sum_{p=0}^{\infty} \pi p \; \varphi_p \, w_p \end{array} \right. \tag{1.12}$$

where w_p is the basis of $L^2(\,]-1,0[\,)$ given by

$$w_0(\widehat{\nu}) = 1, \quad w_p(\widehat{\nu}) = \sqrt{2}\, \cos p\pi\widehat{\nu}, \quad p = 1,2,3,\cdots \tag{1.13}$$

Proposition 1.4. *The problem (1.11,1.10) is well posed and admits a unique solution.*

Proof. We introduce the following closed subspace of $\mathrm{H}^1(\widehat{J}_\alpha)$ defined as

$$\mathcal{V} = \left\{ \Psi_\alpha \in \mathrm{H}^1(\widehat{J}_\alpha) \text{ such that } \int_{\widehat{J}_\alpha} \Psi_\alpha = 0 \right\}$$

We write the problem onto its variationnal formulation : find $\mathcal{W}_\alpha \in \mathcal{V}$ such that, for all $\Psi_\alpha \in \mathcal{V}$,

$$\int_{\widehat{J}_\alpha} \nabla \mathcal{W}_\alpha \nabla \Psi_\alpha + \int_{\Sigma_\pm^\alpha} \Psi_\alpha T_\pm \mathcal{W}_\alpha = \pm \int_{\Sigma_\pm^\alpha} \Psi_\alpha \tag{1.14}$$

Thanks to the Poincaré-Wirtinger inegality, the seminorm of $\mathrm{H}^1(\widehat{J}_\alpha)$ is a norm on $\mathcal{V}$. Let call $a(\mathcal{W}_\alpha, \Psi_\alpha)$ the left member of (1.14) and $l(\Psi_\alpha)$ the right one. It is easy to see that

$l \in \mathcal{L}(\mathcal{V}, \mathbb{C})$. We use for the bilinear form a the results of the appendix A, in partical the proposition A.5 which ensures the coercivity of a and the proposition A.3 which ensures the continuity of a ■

The function $\mathcal{W}_\alpha$ will appear in the approximate problem through the constant (which is uniquely determined even without (1.10)):

$$K(\alpha) = \int_{\Sigma_+^\alpha} \mathcal{W}_\alpha - \int_{\Sigma_-^\alpha} \mathcal{W}_\alpha \,. \tag{1.15}$$

The new 1D approximate problem that we consider consists in finding $\tilde{u}^\varepsilon :]-L^-, L^+[\to \mathbb{C}$, solution of the 1D Helmholtz equation in each segment $]-L^-,0[$ and $]0,L^+[$ (note the difference with (1.4-(i))):

$$\frac{\partial^2 \tilde{u}^\varepsilon}{\partial s^2} + \omega^2 \tilde{u}^\varepsilon = 0, \qquad \text{in } \;]-L^-,0\,[\,\cup\,]\,0,L^+[\,, \tag{1.16}$$

satisfies the same boundary conditions than u^0 at $s = \pm L^\pm$, i.e. (1.4-(ii)), and finally the transmission conditions

$$\left\{ \begin{array}{lll} [\tilde{u}^\varepsilon] = \varepsilon\, K(\alpha) \left\langle \dfrac{\partial \tilde{u}^\varepsilon}{\partial s} \right\rangle, & \text{where } \left\langle \dfrac{\partial \tilde{u}^\varepsilon}{\partial s} \right\rangle := \dfrac{1}{2}\left(\dfrac{\partial \tilde{u}^\varepsilon}{\partial s}(0^+) + \dfrac{\partial \tilde{u}^\varepsilon}{\partial s}(0^-) \right), \\ \left[\dfrac{\partial \tilde{u}^\varepsilon}{\partial s} \right] = -\varepsilon\, \omega^2 \tan\alpha \; \langle \tilde{u}^\varepsilon \rangle\,, & \text{where } \langle \tilde{u}^\varepsilon \rangle := \frac{1}{2}\left(\tilde{u}^\varepsilon(0^+) + \tilde{u}^\varepsilon(0^-)\right). \end{array} \right. \tag{1.17}$$

The conditions (1.17) are clearly first order modifications of the transmission conditions (1.9). To understand in which sense they provide a better approximation that (1.9), following the definition of $\widehat{u}^0$ from u^0, we reconstruct from $\tilde{u}^\varepsilon$ a 2D solution in the normalized slots as

$$\widehat{u}_\pm^{\varepsilon,app}(s, \widehat{\nu}) = \tilde{u}^\varepsilon(s) \quad \text{ in } \widehat{\Omega}_\pm \tag{1.18}$$

and, for $0 < \delta < \delta^* := \max\{L_-, L_+\}$, we define

$$\widehat{\Omega}_+^\delta =]\delta, L_+[\times]-1,0[\quad \text{and} \quad \widehat{\Omega}_-^\delta =]-L_-,-\delta[\times]-1,0[\tag{1.19}$$

Theorem 1.5. *For any $\varepsilon > 0$, the boundary value problem (1.16, 1.4-(ii), 1.17) is well posed in $\mathrm{H}^1(\,]-L^-,0\,[\,\cup\,]\,0,L^+[\,)$. Moreover, for any $0 < \delta < \delta^*$, there exists a constant C_δ, independent of ε such that*

$$\sum_\pm \|\widehat{u}_\pm^{\varepsilon,app} - \widehat{u}_\pm^\varepsilon\|_{\mathrm{H}^1(\widehat{\Omega}_\pm^\delta)} \;\leqslant\; C_\delta\, \varepsilon^3 \tag{1.20}$$

Remark 1.6. Contrary to what happens in theorem 1.3, it is not possible to take $\delta = 0$ in the error estimate (1.20) (in other words, the constant C_δ blows up when $\delta \to 0$). This is due to the apparition of some boundary layer in the neighborhood of the junction, with an amplitude which is like ε. This will be detailled more carefully in the section 3.2

This theorem will be proved in the section 3.2.

2 The formal expansion

As we said in the introduction, as the problem is multiscale, it is not possible to write a uniform expansion for the solution everywhere in the domain $\Omega_\alpha^\varepsilon$. The method of the matched asymptotic expansions will lead us, we have to consider three distinct zones, respectively two slots zones and a junction zone, in which different expansions will be obtained. However, contrarily to the native intuition, this domain decomposition does not correspond to the partition (1.1) of $\Omega_\alpha^\varepsilon$: in the method of the matched asymptotics, the different domains must overlap, the idea being that the different expansions must "coincide" in the overlaping zones.

2.1 An overlaping domain decomposition

This domain decomposition is valid for the time harmonic case and for the time domain case.
In the following, we will denote $\mathcal{C}$ the class of positive continuous fonctions of $\varepsilon > 0$ that tehd to 0 when $\varepsilon \to 0$, less rapidly than $\varepsilon|\ln(\varepsilon)|$ (a typical example is ε^β, with $\beta < 1$).

$$\mathcal{C} = \left\{ \varphi : \mathbb{R}_+^* \to \mathbb{R}_+^* \ / \ \lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0 \text{ and } \lim_{\varepsilon \to 0} \frac{\varphi(\varepsilon)}{\varepsilon|\ln(\varepsilon)|} = +\infty \right\} \tag{2.1}$$

Given φ in $\mathcal{C}$, we define the two slots as

$$\Omega_\pm(\varepsilon) = \left\{ \mathbf{x} \in \Omega_\alpha^\varepsilon \text{ such that } \varphi(\varepsilon) < \pm \mathbf{x} \cdot \mathbf{t}^\pm < L_\pm \right\} \quad \left(\subset \Omega_\pm^\varepsilon \right) \tag{2.2}$$

and we define the junction slot as

$$J_\alpha(\varepsilon) = \varepsilon \widehat{J}_\alpha \cup \bigcup_\pm \left\{ \mathbf{x} \in \Omega_\alpha^\varepsilon \text{ such that } 0 \leqslant \pm \mathbf{x} \cdot \mathbf{t}^\pm < 2\varphi(\varepsilon) \right\} \quad (\supset J_\alpha^\varepsilon) \tag{2.3}$$

in such a way that we have $\Omega_\alpha^\varepsilon = \Omega_-(\varepsilon) \cup J_\alpha(\varepsilon) \cup \Omega_+(\varepsilon)$ with two overlaping zones (see the figure 2.2)

$$\mathcal{O}_\pm(\varepsilon) = \left\{ \mathbf{x} \in \Omega_\alpha^\varepsilon \text{ such that } \varphi(\varepsilon) < \pm \mathbf{x} \cdot \mathbf{t}^\pm < 2\varphi(\varepsilon) \right\} \tag{2.4}$$

Mapping on the overlaping decomposition : we use here different mappings for the slots and the junction. For the slots, the mapping

$$\mathbf{x} \mapsto (s, \widehat{\nu}) = (\mathbf{x} \cdot \mathbf{t}^\pm, \mathbf{x} \cdot \mathbf{n}^\pm / \varepsilon) \tag{2.5}$$

maps the domains $\Omega_\pm(\varepsilon)$ into the rectangles $\widehat{\Omega}_\pm(\varepsilon)$ with

$$\widehat{\Omega}_+(\varepsilon) =]\varphi(\varepsilon), L^+[\times] - 1, 0[\,, \quad \widehat{\Omega}_-(\varepsilon) =] - L^-, -\varphi(\varepsilon)[\times] - 1, 0[\,,$$

Note that the sets $\widehat{\Omega}_\pm(\varepsilon)$ increase when ε decreases and converge to $\widehat{\Omega}_\pm$ when ε tends to 0.
In the same way, the mapping

$$\mathbf{x} \mapsto \widehat{\mathbf{x}} = \mathbf{x}/\varepsilon \tag{2.6}$$

maps $J_\alpha(\varepsilon)$ onto $\widehat{J}_\alpha(\varepsilon)$ (see the figure 2.1), a domain which increases when ε decreases and converges to the unbounded domain

$$\widehat{J}_{\alpha,\infty} = \widehat{J}_\alpha \cup \widehat{B}_+ \cup \widehat{B}_- \tag{2.7}$$

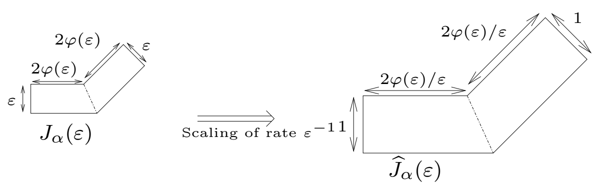

Figure 2.1: The near-field zone

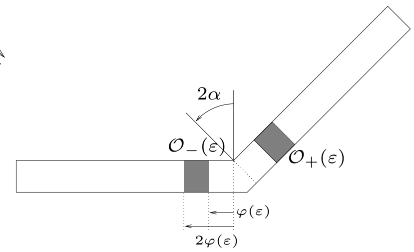

Figure 2.2: The overlaping zones $\mathcal{O}_\pm(\varepsilon)$

2.2 Local expansions and basic equations

We formulate our ansatz for the asymptotic expansions which consists, in each zone after scaling ((2.5) or (2.6)), in looking for power series expansions with respect $\varepsilon\omega$ (for the time harmonic case). In other words, we look for functions

$$u_\pm^k : \widehat{\Omega}_\pm \to \mathbb{C} \quad \text{and} \quad U^k : \widehat{J}_{\alpha,\infty} \to \mathbb{C},\ k \in \mathbb{N}$$

independant of ε such that, at least formally,

$$u^\varepsilon(s, \varepsilon\widehat{\nu}) = \sum_{k=0}^{\infty} (\varepsilon\omega)^k\, u_\pm^k(s, \widehat{\nu}) + o(\varepsilon\omega)^\infty, \qquad \text{in } \widehat{\Omega}_\pm(\varepsilon), \tag{2.8}$$

$$u^\varepsilon(\varepsilon\widehat{\mathbf{x}}) = \sum_{k=0}^{\infty} (\varepsilon\omega)^k\, U^k(\widehat{\mathbf{x}}) + o(\varepsilon\omega)^\infty, \qquad \text{in } \widehat{J}_\alpha(\varepsilon). \tag{2.9}$$

It remains to obtain the equations that will determine the functions $u_\pm^k$ and U^k. For the $u_\pm^k$'s, we substitute formally the expansion (2.8) in the 2D Helmholtz equation written in $\Omega_\pm(\varepsilon)$, using the scaled coordinates $(s, \widehat{\nu})$, and we identify the terms with the same power of ε (since we supposed that our functions do not depend on ε). Straightforward manipulations lead to :

$$\frac{\partial^2 u_\pm^0}{\partial \widehat{\nu}^2} = 0, \quad \frac{\partial^2 u_\pm^1}{\partial \widehat{\nu}^2} = 0, \quad \frac{\partial^2 u_\pm^k}{\partial s^2} + \omega^2 u_\pm^k + \frac{\partial^2 u_\pm^{k+2}}{\partial \widehat{\nu}^2} = 0, \quad k \geqslant 0. \tag{2.10}$$

while the Neumann boundary condition along the "lateral" sides of lead to

$$\frac{\partial u_\pm^k}{\partial \widehat{\nu}}(s, -1) = \frac{\partial u_\pm^k}{\partial \widehat{\nu}}(s, 0) = 0, \quad \pm\, s > 0, \quad k \geqslant 0. \tag{2.11}$$

Proposition 2.1. *Let $(u_\pm^k)_{k\in\mathbb{N}}$ a family of functions defined on $\widehat{\Omega}_\pm$ and satisfying (2.10) and (2.11); then*

$$(i) \quad u_\pm^k(s,\nu) = u_\pm^k(s), \qquad (ii) \quad \frac{\partial^2 u_\pm^k}{\partial s^2} + \omega^2 u_\pm^k = 0, \quad \pm\, s \in [0, L^\pm[, \quad k \geqslant 0. \tag{2.12}$$

Proof. These properties are easily established by induction on k. Indeed,for $k = 0, 1$, the first two equations of (2.10) combined with (2.11) show that $u_\pm^0$ and $u_\pm^1$ are independant of $\widehat{\nu}$. Then, integrating the third equation of (2.10) written for $k = 0$ (respectively for $k = 1$) with respect to $\widehat{\nu}$ (from -1 to 0) and using the boundary conditions for $k = 2$ and $k = 3$, we see that $u_\pm^0$ and $u_\pm^1$ satisfy (2.12-(ii)).
Assume that (2.12) holds up to $k = p$. Then the third equation of (2.10) written for $k = p - 1$ combined with (2.11) written for $k = p + 1$ show that $u_\pm^{p+1}$ is independant of $\widehat{\nu}$. Next, integrating the third equation of (2.10) written for $k = p + 1$ combined with (2.11) written for $k = p + 3$ leads to (2.12-(ii)) for $k = p + 1$. ■

Moreover, from the boundary conditions on $\Gamma_\pm$ in (1.3), we deduce

$$(i) \quad \Big(\frac{\partial u_+^k}{\partial s} - \imath\, \omega\, u_+^k\Big)(L_+) = 0, \quad k \geqslant 0, \qquad (ii) \quad \frac{\partial u_-^k}{\partial s}(-L_-) = -f \quad \text{for } k = 0, \quad = 0 \quad \text{for } k \geqslant 1. \tag{2.13}$$

To obtain the equations for the U^k's, we substitute formally the expansion (2.9) in the 2D Helmholtz equation written in $\widehat{J}_\alpha(\varepsilon)$, using the scaled coordinates $\widehat{\mathbf{x}}$, and we identify the terms with the same power of ε. This permits to see that the U^k's satisfy embedded Laplace's equations

$$\Delta U^0 = 0, \quad \Delta U^1 = 0, \quad \Delta U^k + U^{k-2} = 0, \quad k \geqslant 2, \quad \text{in } \widehat{J}_{\alpha,\infty}\,. \tag{2.14}$$

with Neumann boundary conditions

$$\frac{\partial U^k}{\partial n} = 0, \quad \text{on } \partial\widehat{J}_{\alpha,\infty}, \quad k \in \mathbb{N}. \tag{2.15}$$

Remark 2.2. In the sequel, we shall adopt the convention that all quantities super-indexed by k (such as U^k, $u_\pm^k$, ...) are 0 for negative values of k. This will be useful to simplify some formulas. For instance, with this convention the last equation of (2.14) is also valid for $k = 0, 1$.

2.3 Matching conditions

Here, we cannot characterize fully the functions $(u_\pm^k, U^k)$'s: we miss boundary conditions at $s = 0$ for the $u_\pm^k$'s and conditions at infinity for the U^k's. These conditions, that will be given by the matching conditions, will couple the $u_\pm^k$'s and the U^k's. To express these matching conditions, it is useful to describe the form of the functions U^k's in the two semi-strips $\widehat{B}_\pm$ (see figure 2.3): this is the object of the next section.

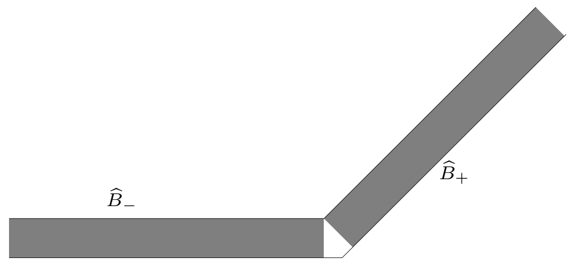

Figure 2.3: The subdomains $\widehat{B}_\pm$

2.3.1 Modal expansion of solutions of embeddeb Laplace equations

Let consider $U^k_\pm : \widehat{B}_\pm \to \mathbb{C} \quad k \geqslant 0$, in $\mathrm{H}^1_{loc}(\widehat{B}_\pm)$ satisfying

$$\Delta U^0_\pm = 0, \quad \Delta U^1_\pm = 0, \quad \Delta U^k_\pm + U^{k-2}_\pm = 0, \quad \text{in } \widehat{B}_\pm, \quad \frac{\partial U^k_\pm}{\partial \widehat{\nu}} = 0 \quad \text{for } \widehat{\nu} = -1, 0, \qquad k \geqslant 2. \tag{2.16}$$

Later, in section 2.3.2, the results of the present section will be applied to the restriction of the U^k's on $\widehat{B}_\pm$, where U^k's are the coefficients in the expansion (2.9).

In $\widehat{B}_\pm$, we shall use the local coordinates $(\widehat{s}, \widehat{\nu}) = (\widehat{\mathbf{x}} \cdot \mathbf{t}^\pm, \widehat{\mathbf{x}} \cdot \mathbf{n}^\pm)$ such that

$$\widehat{\mathbf{x}} \in \widehat{B}_\pm \quad \Longleftrightarrow \quad (\pm\, \widehat{s}, \widehat{\nu}) \in]0, \infty[\times] - 1, 0[. \tag{2.17}$$

The behaviour of the fields U^k in the two semi-strips $\widehat{B}_\pm$ is easily described using separation of variables in $(\widehat{s}, \widehat{\nu})$ coordinates, which naturally introduce the basis w_p (cf (1.13)), which are adapted to the Neumann conditions at $\widehat{\nu} = -1$ or 0 : there exists 1D functions $U^k_{\pm,p}(\widehat{s})$ such that

$$U^k_\pm(\widehat{s}, \widehat{\nu}) = \sum_{p \in \mathbb{N}} U^k_{\pm,p}(\widehat{s}) w_p(\widehat{\nu}) \tag{2.18}$$

If we substitute formally the expression (2.18) into the equations (2.14) (written in the semi-strips $\widehat{B}_\pm$), we obtain

$$\forall\, k \in \{0, 1\},\ \forall p \in \mathbb{N}, \quad \frac{\partial^2 U^k_{\pm,p}}{\partial \widehat{s}^2} - p^2\pi^2 U^k_{\pm,p} = 0 \tag{2.19}$$

$$\forall k \geqslant 2,\ \forall\, p \in \mathbb{N}, \quad \frac{\partial^2 U^k_{\pm,p}}{\partial \widehat{s}^2} - p^2\pi^2 U^k_{\pm,p} + U^{k-2}_{\pm,p} = 0 \tag{2.20}$$

The resolution of (2.19, 2.20) is a tedious but simple exercise on ordinary differential equations. In what follows, we reproduce some results of [JT06], that we present in a slightly different form, more adapted to the purpose of this report. After having remarked that the change of unknown

$$U^k_{\pm,p}(\widehat{s},\widehat{\nu}) = \exp(\pm\pi\widehat{s})V^k_{\pm,p}(\widehat{s},\widehat{\nu})$$

leads to the equations (with the convention $V^k_{\pm,p} \equiv 0$ for $k < 0$)

$$\frac{\partial^2 V^k_{\pm,p}}{\partial \widehat{s}^2} \pm 2p\pi \frac{\partial V^k_{\pm,p}}{\partial \widehat{s}} = -\, V^{k-2}_{\pm,p} \tag{2.21}$$

We introduce, for each $p \in \mathbb{N}$, two sequences of polynomial solutions of (2.21)

$$\big(c^k_{\pm,p}(\widehat{s}), d^k_{\pm,p}(\widehat{s})\big), \quad k \in \mathbb{N},$$

which are defined inductively on k, for each $p \in \mathbb{N}$ and are identically 0 for odd values of k.

- The value $p = 0$ plays a particular role, since equation (2.21) degenerates. For $k =), 1$ one has
$$c^0_{\pm,0}(\widehat{s}) = \imath\, \widehat{s}, \;\; c^1_{\pm,0}(\widehat{s}) = 0, \qquad d^0_{\pm,0}(\widehat{s}) = 1, \;\; d^1_{\pm,0}(\widehat{s}) = 0, \tag{2.22}$$
continuing for $k \geqslant 2$ with:
$$\left| \begin{array}{ll} \dfrac{\partial^2 c^k_{\pm,0}}{\partial \widehat{s}^2} = -\, c^{k-2}_{\pm,0}, & c^k_{\pm,0}(0) = \dfrac{\partial c^k_{\pm,0}}{\partial \widehat{s}}(0) = 0, \\ \dfrac{\partial^2 d^k_{\pm,0}}{\partial \widehat{s}^2} = -\, d^{k-2}_{\pm,0}, & d^k_{\pm,0}(\widehat{0}) = \dfrac{\partial d^k_{\pm,0}}{\partial \widehat{s}}(0) = 0, \end{array} \right. \tag{2.23}$$
It is easy to see that, for even k's, one recovers the monomials of the series expansion of $\exp(\imath\, \widehat{s})$:
$$\widehat{c}^{2m}_{\pm,0}(\widehat{s}) = \frac{(\imath\widehat{s})^{2m+1}}{(2m+1)!}\,, \quad \widehat{d}^{2m}_{\pm,0}(\widehat{s}) = \frac{(\imath\widehat{s})^{2m}}{(2m)!}. \tag{2.24}$$

- For $p \geqslant 1$, one starts from
$$c^0_{\pm,p}(\widehat{s}) = 1, \quad c^1_{\pm,p}(\widehat{s}) = 0 \text{ for } p \geqslant 0; \quad d^0_{\pm,p}(\widehat{s}) = 1, \quad d^1_{\pm,p}(\widehat{s}) = 0. \text{ for } p \geqslant 0, \tag{2.25}$$
Then, $\big(c^k_{\pm,p}(\widehat{s}), d^k_{\pm,p}(\widehat{s})\big)$ are defined as the polynomial solutions of
$$\left| \begin{array}{ll} \dfrac{\partial^2 c^k_{\pm,p}}{\partial \widehat{s}^2} \pm 2p\pi \dfrac{\partial c^k_{\pm,p}}{\partial \widehat{s}} = -\, c^{k-2}_{\pm,p}, & c^k_{\pm,p}(\widehat{0}) = 0, \\ \dfrac{\partial^2 d^k_{\pm,p}}{\partial \widehat{s}^2} \mp 2p\pi \dfrac{\partial d^k_{\pm,p}}{\partial \widehat{s}} = -\, d^{k-2}_{\pm,p}, & d^k_{\pm,p}(\widehat{0}) = 0. \end{array} \right. \tag{2.26}$$
Note the difference between the two lines of (2.26)

Proposition 2.3. *Let* $\left(c^k_{\pm,p}(\widehat{s}), d^k_{\pm,p}(\widehat{s})\right)$ *be a polynomial family of functions satisfying (2.25, 2.26), then :*

- *This family is well-defined an is unique*
- $c^{2m}_{\pm,p}$ *and* $d^{2m}_{\pm,p}$ *have degree* m
- $c^{2m+1}_{\pm,p} = d^{2m+1}_{\pm,p} = 0$
- *We have relations linking* $d^k_{\pm,p}$ *and* $c^k_{\pm,p}$ *:*

$$d^k_{\pm,p}(s) = c^k_{\pm,p}(-s) = d^k_{\mp,p}(-s), \qquad p \geqslant 1, \quad k \geqslant 0 \tag{2.27}$$

- *We have relations linking* $d^k_{\pm,p}$ *and* $d^k_{\pm,1}$ *:*

$$d^k_{\pm,p}(s) = p^{-k}\, d^k_{\pm,1}(p\, s), \qquad p \geqslant 1, \quad k \geqslant 0 \tag{2.28}$$

Proof. We have five points to proove. For the first point, we simply have to study the kernel of the partial differential operator

$$\frac{\partial^2 u}{\partial s^2} \pm 2p\pi \frac{\partial u}{\partial s} \tag{2.29}$$

and prove that there exists a unique polynomial solution of the partial differential

$$\frac{\partial^2 u}{\partial s^2} \pm 2p\pi \frac{\partial u}{\partial s} = s^k, \quad u(0) = 0, \quad k \in \mathbb{N} \tag{2.30}$$

and this solution is exactly of degree $k+1$. Looking for the kernel of (2.29), we can see that there exists two constant $(a, b) \in \mathbb{C}$ such that

$$u(s) = a \exp(\mp 2p\pi s) + b \tag{2.31}$$

We can see then that if we are looking for two polynomial solutions of (2.30) that we call u and v, the difference is a polynomial in the kernel of (2.29), and the expression (2.31) gives immediately that $u - v \equiv 0$, so if (2.30) has a solution, this solution is unique. Now we will exhib this solution : let u be of the form

$$\mathbf{U}^k_{\pm,p}(s) = \sum_{l=1}^{k+1} a^p_{l,k} s^l \tag{2.32}$$

Injecting (2.32) in the left part of (2.30) gives :

$$\sum_{l=0}^{k-1} l + 1 \left(a^p_{l+2,k}(l+2) \pm 2\pi p a^p_{l+1,k}\right) s^l \pm 2\pi p a^p_{k+1,k} s^k = s^k \tag{2.33}$$

This gives immediately that

$$\left|\begin{array}{l} a^p_{k+1,k} = \dfrac{\pm 1}{2\pi p} \\ a^p_{l+1,k} = \dfrac{\mp(l+2)a^p_{l+2,k}}{2\pi p}, \quad l \leqslant k-1 \end{array}\right. \tag{2.34}$$

So the function $\mathbf{U}^k_{\pm,p}$ exists and is solution of (2.30).

By induction on $m \in \mathbb{N}$, using the resolution of (2.30) and using the expression (2.32), the second point is easily proved.

By induction on $m \in \mathbb{N}$ and using the knowledge of the kernel of (2.29), the third point is easily proved.

For the fourth and fifth point, we could use the expression (2.34), but there's a quite more beautiful way. We prove these properties by induction on k. For $k = 0$ and k odd, these properties are clearly true, since $d^0_{\pm,p}$ and $c^0_{\pm,p}$ are equal to 1. Suppose that these properties are true up for $k = 2m$. We apply the operator (2.29) to $d^{m+2}_{\pm,p}(\bullet)$ and (respectively) $c^{m+2}_{\pm,p}(-\bullet)$, $d^{m+2}_{\mp,p}(-\bullet)$ and $p^{-2-m}d^{m+2}_{\pm,1}(p\bullet)$, and after computation, we can see that the difference is in the kernel of this operator, is polynomial and vanishes at $s = 0$. We get immediately that the difference vanishes, and then these properties are true for $k = 2m + 2$; the recurrence is established. ■

Next, we construct two families of functions $\mathbf{c}^k_{\pm,p}$ and $\mathbf{d}^k_{\pm,p}$ from $\widehat{\Omega}_\pm$ into $\mathbb{C}$, for $p \in \mathbb{N}$ and $k \in \mathbb{N}^*$, by:

$$\left|\begin{array}{l} \mathbf{c}^k_{\pm,p}(\widehat{\mathbf{x}}) = \exp\left(\pm p\pi \widehat{s}\right)\, c^k_{\pm,p}(\widehat{s})\, w_p(\widehat{\nu}), \\[2ex] \mathbf{d}^k_{\pm,p}(\widehat{\mathbf{x}}) = \exp\left(\mp p\pi \widehat{s}\right)\, d^k_{\pm,p}(\widehat{s})\, w_p(\widehat{\nu}), \end{array}\right. \tag{2.35}$$

that constitute particular families of embedded Laplace's equations:

$$\forall p \in \mathbb{N}, \quad \forall k \in \mathbb{N}, \quad \Delta \mathbf{c}^k_{\pm,p} = -\mathbf{c}^{k-2}_{\pm,p}, \quad \Delta \mathbf{d}^k_{\pm,p} = -\mathbf{d}^{k-2}_{\pm,p}, \quad \text{in } \widehat{\Omega}_\pm \tag{2.36}$$

with homogeneous Neumann boundary conditions at $\widehat{\nu} = -1, 0$.

Lemma 2.4 (Fundamental lemma for the expansion of $U^k_\pm$)**.** *Let $\left\{U^k_\pm \in H^1_{loc}(\widehat{B}_\pm),\ k \geqslant 0\right\}$ satisfying (2.16), there exist two sequences $(\gamma^\pm_{p,k})_{(p,k)\in\mathbb{N}^2}$ and $(\delta^\pm_{p,k})_{(p,k)\in\mathbb{N}^2}$ of complex numbers such as*

$$U^k_\pm = \sum_{m=0}^{k}\sum_{p=0}^{\infty}\Big(\gamma^\pm_{p,k-m}\, \mathbf{c}^m_{\pm,p} + \delta^\pm_{p,k-m}\, \mathbf{d}^m_{\pm,p}\Big), \quad (\textit{in } \mathrm{H}^1_{loc}(\widehat{B}_\pm)). \tag{2.37}$$

Proof. First, note that the result is true for $k = 0$ and $k = 1$ (harmonic functions in $\widehat{B}_\pm$ that satisfy the Neumann conditions at $\widehat{\nu} = -1, 0$ are linear combinations of the $\mathbf{c}^0_{\pm,p}$'s and

$\mathbf{d}^0_{\pm,p}$'s) . By induction, let us admit that the sequences $\gamma^{\pm}_{p,l}$ and $\delta^{\pm}_{p,l}$ have been constructed up to $l = k-1$. Let us introduce (remember that $\mathbf{c}^1_{\pm,p} = \mathbf{d}^1_{\pm,p} = 0$) :

$$U^{k,*}_{\pm} = \sum_{m=1}^{k}\sum_{p=0}^{\infty}\Big(\gamma^{\pm}_{p,k-m}\ \mathbf{c}^m_{\pm,p} + \delta^{\pm}_{p,k-m}\ \mathbf{d}^m_{\pm,p}\Big) = \sum_{m=2}^{k}\sum_{p=0}^{\infty}\Big(\gamma^{\pm}_{p,k-m}\ \mathbf{c}^m_{\pm,p} + \delta^{\pm}_{p,k-m}\mathbf{d}^m_{\pm,p}\Big)$$

Then we get (using (2.36) and applying the change of index $m \to m-2$) :

$$\begin{aligned}
\Delta U^{k,*}_{\pm} &= \sum_{m=2}^{k}\sum_{p=0}^{\infty}\Big(\gamma^{\pm}_{p,k-m}\Delta\mathbf{c}^m_{\pm,p} + \delta^{\pm}_{p,k-m}\Delta\mathbf{d}^m_{\pm,p}\Big)\\
&= \sum_{m=2}^{k}\sum_{p=0}^{\infty}\Big(\gamma^{\pm}_{p,k-m}\ \mathbf{c}^{m-2}_{\pm,p} + \delta^{\pm}_{p,k-m}\ \mathbf{d}^{m-2}_{\pm,p}\Big)\\
&= \sum_{m=0}^{k-2}\sum_{p=0}^{\infty}\Big(\gamma^{\pm}_{p,k-2-m}\ \mathbf{c}^m_{\pm,p} + \delta^{\pm}_{p,k-2-m}\ \mathbf{d}^m_{\pm,p}\Big) \ = \ -U^{k-2}_{\pm}
\end{aligned}$$

Thus, the function $U^k_{\pm} - U^{k,*}_{\pm}$ is harmonic on $\widehat{B}_{\pm}$, and we know that there exist two sequences of complex numbers, that we choose to denote repectively by $(\gamma^{\pm}_{p,k})_{p\in\mathbb{N}}$ and $(\delta^{\pm}_{p,k})_{p\in\mathbb{N}}$, such that

$$U^k_{\pm} - U^{k,*}_{\pm} = \sum_{p=0}^{\infty}\Big(\gamma^{\pm}_{p,k}\ \mathbf{c}^0_{\pm,p} + \delta^{\pm}_{p,k}\ \mathbf{d}^0_{\pm,p}\Big).$$

and the proof is completed. ■

We can formalize this result by introducing the vector spaces

$$\mathcal{V}_k(\widehat{B}_{\pm}) = \text{span}\ \big\{\big(\mathbf{c}^m_{\pm,p}, \mathbf{d}^m_{\pm,p}\big),\quad p\in\mathbb{N},\quad m\leqslant k\big\},\qquad \mathcal{V}(\widehat{B}_{\pm}) = \bigcup_{k=0}^{\infty}\mathcal{V}_k(\widehat{B}_{\pm}) \tag{2.38}$$

and the linear forms $\mathcal{U}_{\pm}\in\mathcal{V}(\widehat{B}_{\pm}) \to \mathcal{N}^{\pm}_{p,m}(\mathcal{U}_{\pm})\in\mathbb{C}$ and $\mathcal{U}_{\pm}\in\mathcal{V}(\widehat{B}_{\pm}) \to \mathcal{D}^{\pm}_{p,m}(\mathcal{U}_{\pm})\in\mathbb{C}$ such that

$$\forall\,\mathcal{U}_{\pm}\in\mathcal{V}_k(\widehat{B}_{\pm}),\quad \mathcal{U}_{\pm} \ = \ \sum_{m=0}^{k}\ \sum_{p=0}^{\infty}\Big(\mathcal{N}^{\pm}_{p,m}(\mathcal{U}_{\pm})\ \mathbf{c}^m_{\pm,p} + \mathcal{D}^{\pm}_{p,m}(\mathcal{U}_{\pm})\ \mathbf{d}^m_{\pm,p}\Big). \tag{2.39}$$

In the following, the value $m=0$ will play a particular role and that is why we shall denote

$$\forall\,\mathcal{U}_{\pm}\in\mathcal{V}(\widehat{B}_{\pm}),\quad \mathcal{N}^{\pm}_{p}(\mathcal{U}_{\pm}) := \mathcal{N}^{\pm}_{p,0}(\mathcal{U}_{\pm}),\quad \mathcal{D}^{\pm}_{p}(\mathcal{U}_{\pm}) := \mathcal{D}^{\pm}_{p,0}(\mathcal{U}_{\pm}). \tag{2.40}$$

With this notation, the lemma 2.4 can be reinterpreted as

Corollary 2.5. *Let* $\left\{U_\pm^k \in H^1_{loc}(\widehat{B}_\pm),\ k \geqslant 0\right\}$ *satisfying (2.16), then*

$$U_\pm^k \in \mathcal{V}_k(\widehat{B}_\pm) \quad \textit{and} \quad \mathcal{N}^\pm_{p,m}(U_\pm^k) = \mathcal{N}^\pm_p(U_\pm^{k-m}), \quad \mathcal{D}^\pm_{p,m}(U_\pm^k) = \mathcal{D}^\pm_p(U_\pm^{k-m}). \tag{2.41}$$

and consequently,

$$U_\pm^k = \sum_{m=0}^{k} \sum_{p=0}^{\infty} \left(\mathcal{N}_p^\pm \left(U_\pm^{k-m}\right)\ \mathbf{c}^m_{\pm,p} + \mathcal{D}_p^\pm \left(U_\pm^{k-m}\right)\ \mathbf{d}^m_{\pm,p}\right), \quad (\textit{in } \mathrm{H}^1_{loc}(\widehat{B}_\pm)). \tag{2.42}$$

We can give a more tractable definition of $\mathcal{N}_0^\pm$ and $\mathcal{D}_0^\pm$ that can be interpreted as "mean" Dirichlet or Neumann trace operators at $\widehat{s} = 0$:

Lemma 2.6.

$$\forall\, \mathcal{U}_\pm \in \mathcal{V}(\widehat{B}_\pm), \quad \mathcal{D}_0^\pm(\mathcal{U}_\pm) = \int_0^1 \mathcal{U}_\pm(0, \widehat{\nu})\, d\widehat{\nu}, \quad \mathcal{N}_0^\pm(\mathcal{U}_\pm) = \frac{1}{\imath} \int_0^1 \frac{\partial \mathcal{U}_\pm}{\partial \widehat{s}}(0, \widehat{\nu})\, d\widehat{\nu}, \tag{2.43}$$

Proof. Integrating (2.39) with respect to $\widehat{\nu}$ gives:

$$\forall\, \mathcal{U}_\pm \in \mathcal{V}_k(\widehat{B}_\pm), \quad \int_0^1 \mathcal{U}_\pm(\widehat{s}, \widehat{\nu})\, d\widehat{\nu} \ = \ \sum_{m=0}^{k} \left(\mathcal{N}^\pm_{0,m}(\mathcal{U}_\pm)\ c^m_{\pm,0}(\widehat{s}) + \mathcal{D}^\pm_{0,m}(\mathcal{U}^\pm)\ d^m_{\pm,0}(\widehat{s})\right). \tag{2.44}$$

For $\widehat{s} = 0$, this equality leads to the first equality of (2.43), thanks to (2.22), (2.23) and (2.40). For the second equality, we first differentiate (2.44) with respect to $\widehat{s}$ and take $\widehat{s} = 0$ (using again (2.22), (2.23) and (2.40)). ■

Moreover, for functions which belong to the kernel of the linear forms $\mathcal{N}_p^\pm$ for $p > 1$, we have

Lemma 2.7. *Let* $\mathcal{V}_0(\widehat{B}_\pm) := \left\{\mathcal{U}_\pm \in \mathcal{V}(\widehat{B}_\pm) \ /\ \forall\, p > 1, \quad \mathcal{N}_p^\pm(\mathcal{U}_\pm) = 0\right\}$. *Then*

$$\forall\, \mathcal{U} \in \mathcal{V}_0(\widehat{B}_\pm), \quad \mathcal{D}_p^\pm(\mathcal{U}_\pm) = \int_0^1 \mathcal{U}_\pm(0, \widehat{\nu})\, w_p(\widehat{\nu})\, d\widehat{\nu} \tag{2.45}$$

Proof. It is similar to the proof of lemma 2.6, except that we first multiply (2.39) by w_p before integrating with respect to $\widehat{\nu}$ and then use (2.25) and (2.26). The details are left to the reader. ■

2.3.2 Derivation of the matching conditions

To derive the matching conditions, we simply write that the two expansions (2.8) and (2.9) must coincide (at least formally) in the overlaping zones (2.4), i.e. denoting $U_\pm^k$ the restriction of U^k to $\widehat{B}_\pm$:

$$\sum_{k=0}^{\infty} (\varepsilon\omega)^k\, U_\pm^k(s/\varepsilon, \widehat{\nu}/\varepsilon) + o(\varepsilon\omega)^\infty = \sum_{k=0}^{\infty} (\varepsilon\omega)^k u_\pm^k(s, \widehat{\nu}/\varepsilon) + o(\varepsilon\omega)^\infty \quad \text{in } \mathcal{O}^\pm(\varepsilon). \tag{2.46}$$

We denote respectively $\mathcal{L}$ and $\mathcal{R}$ the left and right hand side of (2.46). To get another expression for $\mathcal{R}$, we use the Taylor series expansion (in s) of each $u^k_\pm$

$$\mathcal{R} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} (\varepsilon\omega)^k s^m \frac{1}{m!}\frac{\partial^m u^k_\pm}{\partial s^m}(0) + o(\varepsilon\omega)^\infty \tag{2.47}$$

For $\mathcal{L}$, we use the expansion (2.42) for $U^k_\pm$ and obtain (using $\displaystyle\sum_{k=0}^{\infty}\sum_{m=0}^{k} \equiv \sum_{m=0}^{\infty}\sum_{k=m}^{\infty}$)

$$\mathcal{L} = \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{k=m}^{\infty} (\varepsilon\omega)^k \left(\mathcal{N}^\pm_p(U^{k-m}_\pm)\, \mathbf{c}^m_{\pm,p}(s/\varepsilon,\widehat{\nu}/\varepsilon) + \mathcal{D}^\pm_p(U^{k-m}_\pm)\, \mathbf{d}^m_{\pm,p}(s/\varepsilon,\widehat{\nu}/\varepsilon)\right) + o(\varepsilon\omega)^\infty$$

Since for $p \geqslant 1$, the functions $\mathbf{d}^m_{\pm,p}$ are exponentially decreasing when at infinity and since the functions φ belong to the class $\mathcal{C}$ (see (2.1)), the corresponding terms in the previous sum can be "put into" the $o(\varepsilon\omega)^\infty$ part. For the rest of the sum, we distinguish the terms for $p = 0$, for which we use the formulas (2.24) from the terms corresponding to $p \geqslant 1$ (which are exponentially increasing at infinity):

$$\begin{aligned}\mathcal{L} \;=\; & \sum_{m'=0}^{\infty}\sum_{k=2m'}^{\infty} (\varepsilon\omega)^k\, \mathcal{N}^\pm_0(U^{k-2m'}_\pm)\, \frac{(\imath s/\varepsilon)^{2m'+1}}{(2m'+1)!} + \sum_{m'=0}^{\infty}\sum_{k=2m'}^{\infty} (\varepsilon\omega)^k\, \mathcal{D}^\pm_0(U^{k-2m'}_\pm)\, \frac{(\imath s/\varepsilon)^{2m'}}{(2m')!} \\ & + \sum_{m=0}^{\infty}\sum_{p=1}^{\infty}\sum_{k=m}^{\infty} (\varepsilon\omega)^k\, \mathcal{N}^\pm_p(U^{k-m}_\pm)\, \mathbf{c}^m_{\pm,p}(s/\varepsilon,\widehat{\nu}/\varepsilon) + o(\varepsilon\omega)^\infty.\end{aligned}$$

Using the change of index $k \to k+2m'+1$ in the first sum (resp. $k \to k+2m'$ in the second one), we get

$$\begin{aligned}\mathcal{L} \;=\; & \sum_{m'=0}^{\infty}\sum_{k=-1}^{\infty} (\varepsilon\omega)^k \mathcal{N}^\pm_0(U^{k+1}_\pm)\frac{(\imath\omega s)^{2m'+1}}{(2m'+1)!} + \sum_{m'=0}^{\infty}\sum_{k=0}^{\infty}(\varepsilon\omega)^k\mathcal{D}^\pm_0(U_k)\frac{(\imath s\omega)^{2m'}}{(2m')!} \\ & + \sum_{m=0}^{\infty}\sum_{p=1}^{\infty}\sum_{k=m}^{\infty}(\varepsilon\omega)^k\mathcal{N}^\pm_p(U^{k-m}_\pm)\, \mathbf{c}^m_{\pm,p}(s/\varepsilon,\widehat{\nu}/\varepsilon) + o(\varepsilon\omega)^\infty\end{aligned} \tag{2.48}$$

Finally, the formal identification of the expressions (2.48) and (2.47) in the overlaping zone $\mathcal{O}_\pm(\varepsilon)$, as functions of s and ε, will lead us to our matching conditions.

First, for $p \geqslant 1$, after multiplication of (2.48) and (2.47) by $w_p(\widehat{\nu})$ and integration over $\widehat{\nu}$, we get

$$\sum_{m=0}^{\infty} c^m_{\pm,p}(s/\varepsilon)\left(\sum_{k=m}^{\infty} (\varepsilon\omega)^k\, \mathcal{N}^\pm_p(U^{k-m}_\pm)\right) = 0.$$

The proposition 2.3 implies that the functions $c_{\pm,p}^{2m'}$ are linearly independent and one deduces from that $R_p^c(U_\pm^{k-2m'}) = 0, \forall\, m' \geqslant 0, k \geqslant 0$, that is to say

$$\mathcal{N}_p^{\pm}(U_k) = 0, \quad p \geqslant 1, \quad k \geqslant 0. \tag{2.49}$$

which express the absence of exponentially growing terms. We can see that (2.49) leads to the following condition

$$U^k \text{ grows as most polynomially at infinity in } \widehat{J}_{\alpha,\infty} \tag{2.50}$$

Next, it remains to identify power series expansions. The identification of the terms in $(\varepsilon\omega)^k s^m$, distinguishing even and odd values of m, leads to

$$(\imath\omega)^{2m'}\, \mathcal{D}_0^{\pm}(U_\pm^k) = \frac{\partial^{2m'} u_\pm^k}{\partial s^{2m'}}(0), \quad (\imath\omega)^{2m'+1}\, \mathcal{N}_0^{\pm}(U_\pm^k) = \frac{\partial^{2m'+1} u_\pm^{k-1}}{\partial s^{2m'+1}}(0), \quad m' \geqslant 0, \quad k \geqslant 0.$$

Using the fact that each $u_\pm^k$ solves the 1D Helmholtz equation, we have

$$\frac{1}{(\imath\omega)^{2m'}}\, \frac{\partial^{2m'} u_\pm^k}{\partial s^{2m'}} = u_\pm^k,$$

and using lemma 2.6, we get the "Dirichlet" and "Neumann" matching conditions, namely

$$\left\{\begin{array}{lll} (\mathcal{D}) & u_\pm^k(0) = \displaystyle\int_0^1 U_\pm^k(\widehat{s},\widehat{\nu})\, d\widehat{\nu}, & k \geqslant 0, \\[2ex] (\mathcal{N}) & \dfrac{\partial u_\pm^{k-1}}{\partial s}(0) = \omega \displaystyle\int_0^1 \frac{\partial U_\pm^k}{\partial \widehat{s}}(0,\widehat{\nu})\, d\widehat{\nu}, & k \geqslant 0. \end{array}\right. \tag{2.51}$$

3 Justification of the formal expansion and error estimates

In this paragraph, our goal is to prove that the functions (u_+^k, u_-^k, U^k) are uniquely defined, and that there exists an approximate function built from these functions which differs from the solution of the exact problem with some power of ε that is increasing with the order of the approximation we consider.

3.1 Existence and uniqueness of the formal expansion

In this section, our goal is to prove that the equations (2.12, 2.13, 2.14, 2.15) together the matching conditions (2.51) - define a unique family

$$\Big\{ (u_+^k, u_-^k, U^k) \in \mathrm{H}^1(\widehat{\Omega}_+) \times \mathrm{H}^1(\widehat{\Omega}_-) \times \mathrm{H}^1_{loc}(\widehat{J}_{\alpha,\infty}), \quad k \geqslant 0 \Big\}$$

.

To reach this goal, we first formulate an equivalent problem where the unknowns $(U_k)_{k\in\mathbb{N}}$ are restricted to the junction $\widehat{J}_\alpha$ (section 3.1.1), which is moreover useful for numerical computations. Next, we prove the existence-uniqueness result by induction on k (section 3.1.2).

3.1.1 Restriction to a bounded domain of the problems for the U^k

Our goal in this section is to characterize the restrictions of the functions U^k's to the junction $\widehat{J}_\alpha$ by giving exact Dirichlet to Neumann boundary conditions at the interfaces $\Sigma_\pm^\alpha$ (see figure 1.2).

Using (2.42), (2.43), (2.45) and (2.49), we can write, separating $m = 0$ from $m \geqslant 1$:

$$\left|\begin{array}{lll} U_\pm^k & = & \dfrac{1}{\imath}\Big[\displaystyle\int_0^1 \frac{\partial U_\pm^k}{\partial \widehat{s}}(0,\widehat{\nu})\, d\widehat{\nu}\Big]\, \mathbf{c}^0_{\pm,0} \quad + \quad \displaystyle\sum_{p=0}^{\infty} \Big[\int_0^1 U_\pm^k(0,\widehat{\nu})\, w_p(\widehat{\nu})\, d\widehat{\nu}\Big]\, \mathbf{d}^0_{\pm,p} \\ & + & \dfrac{1}{\imath}\displaystyle\sum_{m=1}^{k} \Big[\int_0^1 \frac{\partial U_\pm^{k-m}}{\partial \widehat{s}}(0,\widehat{\nu})\, d\widehat{\nu}\Big]\, \mathbf{c}^m_{\pm,0} \quad + \quad \displaystyle\sum_{m=1}^{k}\sum_{p=0}^{\infty} \Big[\int_0^1 U_\pm^{k-m}(0,\widehat{\nu})\, w_p(\widehat{\nu})\, d\widehat{\nu}\Big]\, \mathbf{d}^m_{\pm,p} \end{array}\right. \tag{3.1}$$

To compute the trace of $U_\pm^k$ and its normal derivative of $U_\pm^k$ on $\Sigma_\alpha^\pm$, we remark using (2.24) that

$$\left\{\begin{array}{l} \dfrac{\partial \mathbf{c}^0_{\pm,0}}{\partial \widehat{s}}(0,\widehat{\nu}) = \imath, \quad \dfrac{\partial \mathbf{d}^0_{\pm,0}}{\partial \widehat{s}}(0,\widehat{\nu}) = 0, \\ \dfrac{\partial \mathbf{c}^m_{\pm,0}}{\partial \widehat{s}}(0,\widehat{\nu}) = \dfrac{\partial \mathbf{d}^m_{\pm,0}}{\partial \widehat{s}}(0,\widehat{\nu}) = 0 \qquad \text{for } m \geqslant 1. \end{array}\right. \tag{3.2}$$

Moreover, we can deduce from (2.22) to (2.26) the existence of $\delta_p^m \in \mathbb{C}, p \geqslant 1, m \geqslant 0$ such that

$$\frac{\partial d_{\pm,p}^m}{\partial \widehat{s}}(0) = \pm\, \delta_p^m \quad \text{with moreover} \qquad \delta_p^0 = -p\pi \quad \text{and} \quad \delta_p^{2m+1} = 0. \tag{3.3}$$

and therefore that (differentiate (2.35) with respect to $\widehat{s}$)

$$\frac{\partial \mathbf{d}_{\pm,p}^0}{\partial \widehat{s}}(0,\widehat{\nu}) = \pm\, \delta_p^0\, w_p(\widehat{\nu}) \quad \text{for } p \geqslant 1, \quad k \geqslant 1. \tag{3.4}$$

Then, using (3.2) and (3.4), we deduce from (3.1) that

$$\left\{ \begin{array}{rcl} \dfrac{\partial U_\pm^k}{\partial \widehat{s}}(0,\widehat{\nu}) & = & \Big[\displaystyle\int_0^1 \frac{\partial U_\pm^k}{\partial \widehat{s}}(0,\widehat{\nu})\, d\widehat{\nu} \Big] \quad - \quad \displaystyle\sum_{p=1}^{\infty} p\pi \Big[\int_0^1 U_\pm^k(0,\widehat{\nu})\, w_p(\widehat{\nu})\, d\widehat{\nu} \Big]\, w_p(\widehat{\nu}) \\ & & \quad \pm \quad \displaystyle\sum_{m=1}^{k} \sum_{p=1}^{\infty} \delta_p^m\, \Big[\int_0^1 U_\pm^{k-m}(0,\widehat{\nu})\, w_p(\widehat{\nu})\, d\widehat{\nu} \Big]\, w_p(\widehat{\nu}). \end{array} \right. \tag{3.5}$$

Finally, using the definition of the operator $T_\pm$ (see (1.12)) and the Neumann matching condition (2.51-($\mathcal{N}$)), we see that U^k satisfies the non homogeneous DtN condition :

$$\frac{\partial U^k}{\partial n} + T_\pm U^k = \pm \frac{1}{\omega} \frac{\partial u_\pm^{k-1}}{\partial s}(0) \; + \; \sum_{m=1}^{k} \sum_{p=1}^{\infty} \delta_p^m\, \Big[\int_0^1 U_\pm^{k-m}(0,\widehat{\nu})\, w_p(\widehat{\nu})\, d\widehat{\nu} \Big]\, w_p(\widehat{\nu}), \quad \text{on } \Sigma_\pm^\alpha \tag{3.6}$$

Finally, we obtain a problem "equivalent" to (2.12, 2.13, 2.14, 2.15, 2.51) by replacing (2.51-($\mathcal{N}$)) by the DtN condition (3.6). The precise statement is the following :

Theorem 3.1. *Let* $\Big\{ (u_+^k, u_-^k, U^k) \in \mathrm{H}^1(\widehat{\Omega}_+) \times \mathrm{H}^1(\widehat{\Omega}_-) \times \mathrm{H}^1_{loc}(\widehat{J}_{\alpha,\infty}), \quad k \geqslant 0 \Big\}$ *be a solution of (2.12, 2.13, 2.14, 2.15) with the matching conditions (2.51), then*

$$\Big\{ (u_+^k, u_-^k, U^k) \in \mathrm{H}^1(\widehat{\Omega}_+) \times \mathrm{H}^1(\widehat{\Omega}_-) \times \mathrm{H}^1_{loc}(\widehat{J}_\alpha), \quad k \geqslant 0 \Big\}$$

is solution of (2.12, 2.13), $(2.14, 2.15)_{\widehat{J}_\alpha}$, (2.51-($\mathcal{D}$)) and (3.6), where $(2.14, 2.15)_{\widehat{J}_\alpha}$ holds the restriction of (2.14, 2.15) respectively to $\widehat{J}_\alpha$ and $\partial \widehat{J}_{\alpha,\infty} \cap \partial \widehat{J}_\alpha$.

Reciprocally, if $\Big\{ (u_+^k, u_-^k, U^k) \in \mathrm{H}^1(\widehat{\Omega}_+) \times \mathrm{H}^1(\widehat{\Omega}_-) \times \mathrm{H}^1_{loc}(\widehat{J}_\alpha), \quad k \geqslant 0 \Big\}$ *is solution of (2.12, 2.13), $(2.14, 2.15)_{\widehat{J}_\alpha}$, (2.51-($\mathcal{D}$)) and (3.6), then by extending U^k to $\widehat{B}_\pm$ via (3.1),*

$$\Big\{ (u_+^k, u_-^k, U^k) \in \mathrm{H}^1(\widehat{\Omega}_+) \times \mathrm{H}^1(\widehat{\Omega}_-) \times \mathrm{H}^1_{loc}(\widehat{J}_{\alpha,\infty}), \quad k \geqslant 0 \Big\}$$

is a solution of (2.12, 2.13, 2.14, 2.15) and (2.51).

Proof. The direct statement has been proved. For the reciprocal, let us consider

$$\Big\{(u_+^k, u_-^k, U^k) \in \mathrm{H}^1(\widehat{\Omega}_+) \times \mathrm{H}^1(\widehat{\Omega}_-) \times \mathrm{H}^1_{loc}(\widehat{J}_\alpha), \quad k \geqslant 0\Big\}$$

solution of (2.12, 2.13), $(2.14, 2.15)_{\widehat{J}_\alpha}$, (2.51-($\mathcal{D}$)) and (3.6), and let us extend U^k to $\widehat{B}_\pm$ via (3.1). We have several points to prove :

(i) $U^k \in \mathrm{H}^1_{loc}(\widehat{J}_{\alpha,\infty})$,

(ii) U^k satisfies (2.14, 2.15),

(iii) (2.51-($\mathcal{N}$)) is satisfied.

(i) This point is the easiest one, since the expansion (3.1) is built to satisfy the continuity of the trace of U^k and $\frac{\partial U^k}{\partial n}$ over $\Sigma_\pm^\alpha$

(ii) This point is also easy to prove. We know that (2.14, 2.15) is proved on $\widehat{J}_\alpha \times \Big(\partial\widehat{J}_\alpha \cap \partial\widehat{J}_{\alpha,\infty}\Big)$. We simply have to prove that these equations are also true on $\widehat{B}_\pm \times \Big(\partial\widehat{B}_\pm \cap \partial\widehat{J}_{\alpha,\infty}\Big)$. Note that the proof is very similar (to not say identical) to the proof of the fundamental lemma 2.4. We start from (3.1) :

$$U_\pm^k = \frac{1}{\imath}\sum_{m=0}^{k}\left[\int_0^1 \frac{\partial U_\pm^{k-m}}{\partial\widehat{s}}(0,\widehat{\nu})w_p(\widehat{\nu})d\widehat{\nu}\right]c_{\pm,0}^m + \sum_{m=0}^{k}\sum_{p=0}^{\infty}\left[\int_0^1 U_\pm^{k-m}(0,\widehat{\nu})w_p(\widehat{\nu})d\widehat{\nu}\right]d_{\pm,0}^m \tag{3.7}$$

We apply the Laplacian operator on (3.7), we use (2.36) and after a sum reindexation, we have that

$$\Delta U_\pm^k = U_\pm^{k-2}$$

(iii) Here, there's a little more work to do. We start from (3.1), and we can write easily (3.5), because the work has already be done. The relation (3.5) can be written as

$$\begin{aligned} \frac{\partial U^k}{\partial n} + T_\pm U^k \;&=\; \int_0^1 \frac{\partial U_\pm^k}{\partial n}(0,\widehat{\nu})d\widehat{\nu} \\ &+\; \sum_{m=1}^{k}\sum_{p=1}^{\infty} \delta_p^m \, \Big[\int_0^1 U_\pm^{k-m}(0,\widehat{\nu})\; w_p(\widehat{\nu})\; d\widehat{\nu}\,\Big]\; w_p(\widehat{\nu}), \quad \text{on } \Sigma_\pm^\alpha \end{aligned} \tag{3.8}$$

However, (3.6) must be satisfied. Taking the difference between (3.6) and (3.8) leads to (2.51-($\mathcal{N}$)).

■

3.1.2 Existence and uniqueness

It is done by induction on k. According to theorem 3.1, it suffices to consider the problem (2.12, 2.13), (2.14, 2.15)$_{\widehat{J}_\alpha}$, (3.6) and (2.51-($\mathcal{D}$)). To clearly identify the recursion process, it is useful to reformulate this problem in a more decoupled way (we mean between $u_\pm^k$ and U^k, at each step k), which is also useful from the computational point of view.

To achieve such a decoupling the idea is first to consider (3.6) as a boundary condition for $U_\pm^k$, next to formulate a 1D transmission problem for $u_\pm^k$. That is why we shall use the following two technical lemmas

Lemma 3.2. *Given $\Phi \in \mathrm{L}^2(\widehat{J}_\alpha)$ and $g_\pm \in \mathrm{H}^{-\frac{1}{2}}(\Sigma_\pm^\alpha)$, there exists $U \in \mathrm{H}^1(\widehat{J}_\alpha)$, which is unique up to an additive constant, such that there exists $U \in \mathrm{H}^1(\widehat{J}_\alpha)$, which is unique up to an additive constant, such that*

$$\left\{ \begin{array}{lll} \Delta U = \Phi, & \text{in } \widehat{J}_\alpha \ , \quad \dfrac{\partial U}{\partial n} = 0, & \text{on } \partial \widehat{J}_\alpha \setminus \left(\Sigma_+^\alpha \cap \Sigma_-^\alpha\right). \\ \dfrac{\partial U}{\partial n} + T_\pm U = g_\pm \ , & \text{on } \Sigma_\pm^\alpha \ , & \end{array} \right. \tag{3.9}$$

if and only if one satisfies the compatibility condition

$$\int_{\Sigma_-^\alpha} g_- + \int_{\Sigma_+^\alpha} g_+ = \int_{\widehat{J}_\alpha} \Phi. \tag{3.10}$$

Moreover, $\mathcal{W}_\alpha$ being defined by (1.11, 1.10) , any solution of (3.9) satisfies

$$\int_{\Sigma_+^\alpha} U - \int_{\Sigma_-^\alpha} U = \int_{\widehat{J}_\alpha} \Phi \, \mathcal{W}_\alpha + \int_{\Sigma_-^\alpha} g_- \, \mathcal{W}_\alpha + \int_{\Sigma_+^\alpha} g_+ \, \mathcal{W}_\alpha \ . \tag{3.11}$$

Proof. The existence-uniqueness proof is a classical exercise about Lax-Milgram's lemma and Poincaré-Wirtinger's inequality (the important point is that $T_\pm : \mathrm{H}^{\frac{1}{2}}(\Sigma_\pm^\alpha) \mapsto \mathrm{H}^{-\frac{1}{2}}(\Sigma_\pm^\alpha)$ is a positive symmetric operator whose kernel is the space of constant functions - see the appendix A). The compatibility condition (3.10) is obtained by integrating the first equation of (3.9), using Green's formula and the symmetry of $T_\pm$.

To obtain (3.11), we multiply the equation for U by $\mathcal{W}_\alpha$ and integrate over $\widehat{J}_\alpha$. Using Green's formula twice, and the fact that $\mathcal{W}_\alpha$ is harmonic, we get

$$\sum_\pm \int_{\Sigma_\pm^\alpha} \left(\frac{\partial \mathcal{W}_\alpha}{\partial n} \, U - \frac{\partial U}{\partial n} \, \mathcal{W}_\alpha \right) = \int_{\widehat{J}_\alpha} \Phi \, \mathcal{W}_\alpha$$

Using the boundary conditions on $\Sigma_\pm^\alpha$ for U and $\mathcal{W}_\alpha$ together with the symmetry of $T_\pm$, we obtain (3.10). ∎

Lemma 3.3. *Given* $(j_d, j_n) \in \mathbb{C}\times\mathbb{C}$ *and* $f_- \in \mathbb{C}$ *, there exists a unique* $u_\pm \in \mathrm{H}^1(\,(0,\ \pm L_\pm)\,)$*, such that:*

$$\begin{cases} \dfrac{\partial^2 u_\pm}{\partial s^2} + \omega^2 u_\pm = 0, & \pm\, s \in [0, L^\pm[\ , \\[2ex] \Big(\dfrac{\partial u_+}{\partial s} + \imath\,\omega\, u_+\Big)(L_+) = 0, & \dfrac{\partial u_-}{\partial s}(L_-) = f_-\ , \\[2ex] u_+(0) - u_-(0) = j_d\ , & \dfrac{\partial u_+}{\partial s}(0) - \dfrac{\partial u_-}{\partial s}(0) = j_n\ . \end{cases} \tag{3.12}$$

Proof. The result is straightforward (one can compute explicitely the solution of the problem (3.12)). ■

According to $(2.14,\ 2.15)_{\widehat{\mathcal{J}}_\alpha}$ and (3.6), we can apply lemma 3.2 with $U = U^k$, $\Phi = -U^{k-2}$ and $g_\pm = g_\pm^{k-1}$ with

$$g_\pm^{k-1} := \pm\frac{1}{\omega}\frac{\partial u_\pm^{k-1}}{\partial s}(0)\ +\ \sum_{m=1}^{k}\sum_{p=1}^{\infty} \delta_p^m\ \Big[\int_0^1 U_\pm^{k-m}(0,\widehat{\nu})\ w_p(\widehat{\nu})\ d\widehat{\nu}\,\Big]\ w_p(\widehat{\nu}), \quad \text{on } \Sigma_\pm^\alpha \tag{3.13}$$

where the index $k-1$ in $g_\pm^{k-1}$ is "justified" by the fact that $g_\pm^{k-1}$ is known explicitly when $u_\pm^{k-1}$ and the U^m's for $m \leqslant k-1$ are known. Writing (3.10) gives

$$\frac{1}{\omega}\Big(\frac{\partial u_\pm^{k-1}}{\partial s}(0^+) - \frac{\partial u_\pm^{k-1}}{\partial s}(0^-)\Big) = -\int_{\widehat{\mathcal{J}}_\alpha} U^{k-2}\ .$$

which, written for "$k = k+1$", gives the Neumann jump condition

$$\frac{1}{\omega}\Big(\frac{\partial u_\pm^{k}}{\partial s}(0^+) - \frac{\partial u_\pm^{k}}{\partial s}(0^-)\Big) = -\int_{\widehat{\mathcal{J}}_\alpha} U^{k-1}\ , \quad k \geqslant 0. \tag{3.14}$$

Writing (3.11) gives the Dirichlet jump condition

$$\int_{\Sigma_+^\alpha} U^k - \int_{\Sigma_-^\alpha} U^k = -\int_{\widehat{\mathcal{J}}_\alpha} U^{k-2}\ \mathcal{W}_\alpha + \int_{\Sigma_-^\alpha} g_-^{k-1}\ \mathcal{W}_\alpha + \int_{\Sigma_+^\alpha} g_+^{k-1}\ \mathcal{W}_\alpha\ . \tag{3.15}$$

that can be rewritten, using the matching conditions (2.51-($\mathcal{D}$)):

$$u_\pm^k(0^+) - u_\pm^k(0^-) = -\int_{\widehat{\mathcal{J}}_\alpha} U^{k-2}\ \mathcal{W}_\alpha + \int_{\Sigma_-^\alpha} g_-^{k-1}\ \mathcal{W}_\alpha + \int_{\Sigma_+^\alpha} g_+^{k-1}\ \mathcal{W}_\alpha\ . \tag{3.16}$$

For each k, we have succeeded to decouple the calculation of $u_\pm^k$ since jump conditions (3.14) and (3.16) written for "$k = k+1$" are sufficient, when associated to equations (2.12, 2.13), to determine $u_\pm^k$ uniquely (lemma 3.3).

As the solution of problem (P) with $\Phi = U^{k-2}$ and $g_\pm = g_\pm^{k-1}$, U^k is defined up to an additive constant. To fix this constant we can use again (2.51-($\mathcal{D}$)) (in a symmetric way with respect to $\pm$) :

$$\frac{1}{2}\left(\int_{\Sigma_+^\alpha} U^k + \int_{\Sigma_-^\alpha} U^k\right) = \frac{1}{2}\left(u_+^k(0^+) + u_-^k(0^-)\right) \tag{3.17}$$

Finally, we obtain an equivalent problem to (2.12, 2.13), (2.14, 2.15)$_{\widehat{\mathcal{J}}_\alpha}$, (3.6), (2.51-($\mathcal{D}$)) by replacing (2.51-($\mathcal{D}$)) by (3.14), (3.16) and (3.17). More precisely

Theorem 3.4. *The following two propositions are equivalent (for the clarity of notation, we omit to mention again the functional setting):*

(i) $\left\{ (u_+^k, u_-^k, U^k),\ k \geqslant 0\right\}$ *is solution of (2.12, 2.13), (2.14, 2.15)*$_{\widehat{\mathcal{J}}_\alpha}$*, (3.6) and (2.51-($\mathcal{D}$)).*

(ii) $\left\{ (u_+^k, u_-^k, U^k),\ k \geqslant 0\right\}$ *is solution of (2.12, 2.13, 3.14, 3.16) and ((2.14, 2.15)*$_{\widehat{\mathcal{J}}_\alpha}$*, 3.6, 3.17), with* $g_\pm^{k-1}$ *defined as (3.13).*

Proof. We just proved the implication $(i) \Longrightarrow (ii)$. We will prove the implication $(ii) \Longrightarrow (i)$. Let $\left\{ (u_+^k, u_-^k, U^k),\ k \geqslant 0\right\}$ a solution of (2.12, 2.13, 3.14, 3.16) and ((2.14, 2.15)$_{\widehat{\mathcal{J}}_\alpha}$, 3.6, 3.17), with $g_\pm^{k-1}$ defined as (3.13). The only point we have to proove is that (2.51-($\mathcal{D}$)) is satsisfied. However, (3.17) is exactly

$$\frac{1}{2}\left(\int_{\Sigma_+^\alpha} U^k + \int_{\Sigma_-^\alpha} U^k\right) = \frac{1}{2}\left(u_+^k(0^+) + u_-^k(0^-)\right), \tag{3.18}$$

and, since we can write (3.15) thanks to the problem satisfied by U_k, and since (3.16) must be satisfied, by taking the difference, we get

$$\int_{\Sigma_+^\alpha} U^k - \int_{\Sigma_-^\alpha} U^k \ = \ u_+^k(0^+) - u_-^k(0^-) \tag{3.19}$$

To conclude, one simply has to see the conditions (2.51-($\mathcal{D}$)) as linear combination of (3.18) and (3.19). ■

Next, we show that the problem (2.12, 2.13, 3.14, 3.16) and ((2.14, 2.15)$_{\widehat{\mathcal{J}}_\alpha}$, 3.6, 3.17), with $g_\pm^{k-1}$ defined as (3.13), admits a unique solution $\left\{ (u_+^k, u_-^k, U^k),\ k \geqslant 0\right\}$, by induction on $k \in \mathbb{N}$.

The case $k = 0$. With the convention of remark 2.2, we see from (2.12, 2.13, 3.14, 3.16) that $u^0_\pm$ is, as expected, the solution of the transmission problem for the 1D Helmholtz problem with the transmission conditions (1.9) with $u^0(0^\pm) \equiv u^0_\pm(0)$), i. e. the "concatenation" of u^0_- and u^0_+ is nothing but u^0 as defined in section 1.2 (see (1.4)).

Moreover, we see from (2.14, 2.15) and (3.6) that U^0 solves (3.9) with $\Phi = 0$ and $g_\pm = 0$, which implies that U^0 is constant. Next, (3.17) gives

$$U^0 = \frac{1}{2}\big(u^0_+(0) + u^0_-(0)\big) = u^0(0).$$

The general case $k \geqslant 1$. Assume that $(u^\ell_+, u^\ell_-, U^\ell)$, $\ell \leqslant k-1$ are known, then, according to theorem 3.4,

- We first determine (u^k_+, u^k_-) as the unique solution of the 1D transmission problem (2.12, 2.13) with the transmission conditions (3.14, 3.16). (cf. lemma 3.3)
- We compute $g^{k-1}_\pm$ thanks to (3.13).
- We determine U^k as the solution, cf. lemma 3.2, of the boundary value problem ((2.14, $2.15)_{\widehat{J}_\alpha}$, 3.6, 3.17). One must of course check the compatibility condition (3.10), which is a consequence of (3.14).

Finally, regrouping the above results with theorems 3.1 and 3.4, we have proven the following theorem

Theorem 3.5. *There exists a unique family*

$$\left\{\ (u^k_+, u^k_-, U^k) \in \mathrm{H}^1(\widehat{\Omega}_+) \times \mathrm{H}^1(\widehat{\Omega}_-) \times \mathrm{H}^1_{loc}(\widehat{J}_{\alpha,\infty}), \quad k \geqslant 0\ \right\}$$

satisfying (2.12, 2.13, 2.14, 2.15), the matching conditions (2.51) and the growth conditions (2.50).

3.2 Error estimates

In this section, we will prove the error estimates of the theorem 1.3 of the section 1, by giving the result in the general case. We will prove these results by two steps :

1. A global estimate on $\widehat{\Omega}_\alpha$ (section 3.2.1)
2. A local estimate on $\widehat{\Omega}^\delta_\pm$ defined by (1.19) - here we will understand why we can' t directly take $\delta = 0$ in (1.20) (section 3.2.2)

3.2.1 Global error estimate

We consider $\varphi \in \mathcal{C}$ (going back to section 2.1) and introduce a cut-off function

$$\Phi_\varepsilon(s) = \Phi\left(\frac{s}{\varphi(\varepsilon)}\right) \quad \text{with} \quad \Phi \in \mathcal{C}_0^\infty(\mathbb{R}_+), \quad \text{supp}\ \Phi \in\ [0,2], \quad \Phi(s) = 1 \text{ in } [0,1]$$

from which we construct a $2D$ cut-off function $\widehat{\Phi}_\varepsilon$ defined on $\Omega_\alpha^\varepsilon$ as (going back to the section 1, we have $\Omega_\alpha^\varepsilon = \varepsilon \widehat{J}_\alpha \cup \Omega_-^\varepsilon \cup \Omega_+^\varepsilon$)

- On $\varepsilon \widehat{J}_\alpha$, $\widehat{\Phi}_\varepsilon = 1$
- On $\Omega_\pm^\varepsilon$, $\widehat{\Phi}_\varepsilon(s,\nu) = \Phi_\varepsilon(\pm s)$

The idea is to consider that the variations of the function $\widehat{\Phi}_\varepsilon$ are compactly supported in the overlaping zones $\mathcal{O}_\pm(\varepsilon)$, of course we'll have to choose the function ϕ.

Next, given $k \in \mathbb{N}$, we propose the following global approximate function (one idea is to choose φ with respect to k)

$$u_{app}^{\varepsilon,k} := (1 - \widehat{\Phi}_\varepsilon) \sum_{m=0}^{k} (\varepsilon\omega)^m u_\pm^m + \widehat{\Phi}_\varepsilon \sum_{m=0}^{k} (\varepsilon\omega)^m U^m(\bullet/\varepsilon) \quad \text{on } \Omega_\alpha^\varepsilon \tag{3.20}$$

Thanks to the stability result proved in the appendix B, we only have to study, for any function $v \in \mathrm{H}^1(\Omega_\alpha^\varepsilon)$, the quantity

$$a^\varepsilon(u^\varepsilon - u_{app}^{\varepsilon,k}, v)$$

where a^ε is the left member of (B.1). Thanks to the fact that each function $u_\pm^k$ satisfies (2.13), and thanks to the fact that each function U^k satisfies (2.15), which leads that the functions u^ε and $u_{app}^{\varepsilon,k}$ satisfy the same boundary conditions on $\partial\Omega_\alpha^\varepsilon$, we have

$$a^\varepsilon(u^\varepsilon - u_{app}^{\varepsilon,k}, v) \quad = \quad \frac{1}{\varepsilon} \int_{\Omega_\alpha^\varepsilon} -\Delta\left(u^\varepsilon - u_{app}^{\varepsilon,k}\right)\overline{v} - \omega^2\left(u^\varepsilon - u_{app}^{\varepsilon,k}\right)\overline{v} \tag{3.21}$$

Since u^ε satisfies the Helmholtz equation, this relation becomes

$$a^\varepsilon(u^\varepsilon - u_{app}^{\varepsilon,k}, v) \quad = \quad \frac{1}{\varepsilon} \int_{\Omega_\alpha^\varepsilon} \Delta u_{app}^{\varepsilon,k}\overline{v} + \omega^2 u_{app}^{\varepsilon,k}\overline{v} \tag{3.22}$$

We inject now the expression of $u^{\varepsilon,k}_{app}$ given by (3.20) into (3.22), and we can see that (3.22) can be written as, since Φ_ε is compactly supported for $s \in [\varphi(\varepsilon), 2\varphi(\varepsilon)]$:

$$\begin{aligned} a^\varepsilon(u^\varepsilon - u^{\varepsilon,k}_{app}, v) \quad = \quad & -\frac{1}{\varepsilon\varphi(\varepsilon)^2}\int_{\varphi(\varepsilon)}^{2\varphi(\varepsilon)}\int_{-\varepsilon}^{0}\Phi''\left(\frac{\pm s}{\varepsilon}\right)\sum_{m=0}^{k}(\varepsilon\omega)^m(u^m_\pm(s) - U^m_\pm(s\varepsilon^{-1}, \nu\varepsilon^{-1}))\overline{v}(s,\nu)d\nu ds \\ \mp \quad & \frac{1}{\varepsilon\varphi(\varepsilon)}\int_{\varphi(\varepsilon)}^{2\varphi(\varepsilon)}\int_{-\varepsilon}^{0}\Phi'\left(\frac{\pm s}{\varepsilon}\right)\sum_{m=0}^{k}(\varepsilon\omega)^m\left(\frac{\partial u^m_\pm}{\partial s}(s) - \frac{1}{\varepsilon}\frac{\partial U^m_\pm}{\partial \widehat{s}}(s\varepsilon^{-1}, \nu\varepsilon^{-1})\right)\overline{v}(s,\nu)d\nu ds \\ + \quad & \frac{1}{\varepsilon}\int_{\Omega^\varepsilon_\alpha}\widehat{\Phi}_\varepsilon\sum_{m=0}^{k}(\varepsilon\omega)^m\left(\frac{1}{\varepsilon^2}\Delta_{\widehat{s},\widehat{\nu}}U^m + U^m\right)\overline{v} \\ + \quad & \frac{1}{\varepsilon}\int_{\Omega^\varepsilon_\alpha}(1-\widehat{\Phi}_\varepsilon)\sum_{m=0}^{k}(\varepsilon\omega)^m\left(\frac{\partial^2 u^m_\pm}{\partial s^2} + \omega^2 u^m_\pm\right)\overline{v} \end{aligned} \tag{3.23}$$

Note that there are two similar terms in (3.23) for each function $u^m_\pm$. We treat each term of this relation to get a upper bound of $|a^\varepsilon(u^\varepsilon - u^{\varepsilon,k}_{app}, v)|$ by a constant multiplied by the H^1 norm of v over $\widehat{\Omega}_\alpha$.

The fourth term of (3.23) is the simpliest to treat : since each function $u^m_\pm$ satisfies (2.12), this term vanishes.

The third term is treated differently, since the functions U^k do not satisfy the Helmholtz equation, but (2.14) : this term is equal to

$$\frac{1}{\varepsilon}\int_{\Omega^\varepsilon_\alpha}\widehat{\Phi}_\varepsilon\left((\varepsilon\omega)^{k-1}U^{k-1} + (\varepsilon\omega)^k U^k\right)\overline{v} \tag{3.24}$$

We separate the integration over $\varepsilon\widehat{J}_\alpha$ from the integration over the sets $\Omega^\varepsilon_\pm$. On $\varepsilon\widehat{J}_\alpha$, we simply use the Cauchy-Schwartz inegality, since $\widehat{\Phi}_\varepsilon = 1$:

$$\begin{aligned} & \frac{1}{\varepsilon}\int_{\varepsilon\widehat{J}_\alpha}\left((\varepsilon\omega)^{k-1}U^{k-1} + (\varepsilon\omega)^k U^k\right)\overline{v} \\ \leqslant \varepsilon^{k-2}\omega^{k-1}\left\|U^{k-1}\right\|_{\mathrm{L}^2(\varepsilon\widehat{J}_\alpha)}\|v\|_{\mathrm{L}^2(\varepsilon\widehat{J}_\alpha)} & + \varepsilon^{k-1}\omega^k\left\|U^k\right\|_{\mathrm{L}^2(\varepsilon\widehat{J}_\alpha)}\|v\|_{\mathrm{L}^2(\varepsilon\widehat{J}_\alpha)} \end{aligned}$$

which gives

$$\begin{aligned} & \frac{1}{\varepsilon}\int_{\varepsilon\widehat{J}_\alpha}\left((\varepsilon\omega)^{k-1}U^{k-1} + (\varepsilon\omega)^k U^k\right)\overline{v} \\ \leqslant \varepsilon^{k}\omega^{k-1}\left\|U^{k-1}\right\|_{\mathrm{L}^2(\widehat{J}_\alpha)}\|v\|_{\mathrm{L}^2(\widehat{J}_\alpha)} & + \varepsilon^{k+1}\omega^k\left\|U^k\right\|_{\mathrm{L}^2(\widehat{J}_\alpha)}\|v\|_{\mathrm{L}^2(\widehat{J}_\alpha)} \end{aligned} \tag{3.25}$$

Remark 3.6. For $k=0$, $U^{-1}=0$, and the term (3.25) is like ε. Note moreover that the quantities $\left\|U^k\right\|_{\mathrm{L}^2(\widehat{J}_\alpha)}$ for any k do not depend on ε

For the integration of (3.24) over $\Omega^\varepsilon_\pm$, we have many points to observe :

- First, the quantity in the integral vanishes for $s > 2\varphi(\varepsilon)$
- Then, $U_\pm^k$ is known and given by (2.42), it grows as $s^k\varepsilon^{-k}$, when $s^k\varepsilon^{-k}$ goes to infty (this is the case, because $\phi \in \mathcal{C}$).

We have the following boundary for the term U^k

$$\frac{1}{\varepsilon}\int_{-\varepsilon}^{0}\int_{0}^{2\varphi(\varepsilon)}(\varepsilon\omega)^k U_\pm^k\left(\frac{\pm s}{\varepsilon},\frac{\nu}{\varepsilon}\right)\overline{v}(\pm s,\nu)dsd\nu \leqslant \frac{1}{\varepsilon}\omega^k 2^k\varphi(\varepsilon)^k C_k\int_{0}^{2\varphi(\varepsilon)}\int_{-\varepsilon}^{0}\overline{v}(\pm s,\nu)d\nu ds \tag{3.26}$$

We use to conclude the following lemma

Lemma 3.7. *For all $v \in \mathrm{H}^1(\widehat{\Omega}_\alpha)$, we have*

$$\frac{1}{\varepsilon}\int_{0}^{2\varphi(\varepsilon)}\int_{-\varepsilon}^{0}\overline{v}(\pm s,\varepsilon\nu)d\nu ds \leqslant \sqrt{2\varphi(\varepsilon)}\,\|v\|_{\mathrm{H}^1(\widehat{\Omega}_\alpha)} \tag{3.27}$$

Proof. Clearly, we have, by using the variable change $\widehat{\nu} = \varepsilon\nu$,

$$\frac{1}{\varepsilon}\int_{0}^{2\varphi(\varepsilon)}\int_{-\varepsilon}^{0}\overline{v}(\pm s,\varepsilon\nu)d\nu ds = \int_{0}^{2\varphi(\varepsilon)}\int_{-1}^{0}\overline{v}(\pm s,\widehat{\nu})d\widehat{\nu}ds$$

Then by using the mean-value inegality, one can write that

$$\int_{0}^{2\varphi(\varepsilon)}\int_{-1}^{0}\overline{v}(\pm s,\widehat{\nu})d\widehat{\nu}ds \leqslant \int_{0}^{2\varphi(\varepsilon)}\int_{-1}^{0}\left(\int_{0}^{s}\left|\frac{\partial v}{\partial s}\right|(\sigma,\widehat{\nu})d\sigma\right)d\widehat{\nu}ds \tag{3.28}$$

which is bounded clearly by

$$\int_{0}^{2\varphi(\varepsilon)}\int_{-1}^{0}\left(\int_{0}^{2\varphi(\varepsilon)}\left|\frac{\partial v}{\partial s}\right|(\sigma,\widehat{\nu})d\sigma\right)d\widehat{\nu}ds \tag{3.29}$$

We use in (3.29) the Cauchy-Schwartz inegality on the integral over σ, and we bound the L^2 norm of the derivate of v over s by the H^1 norm of v, and we get (3.27). ■

Finally, using lemma 3.7 in (3.26) gives

$$\int_{-\varepsilon}^{0}\int_{0}^{2\varphi(\varepsilon)}(\varepsilon\omega)^k U_\pm^k\left(\frac{\pm s}{\varepsilon},\frac{\nu}{\varepsilon}\right)\overline{v}(\pm s,\nu)dsd\nu \leqslant \frac{1}{\varepsilon}\omega^k C_k 2^k\varphi(\varepsilon)^{k+1/2}\,\|v\|_{\mathrm{H}^1(\widehat{\Omega}_\alpha)} \tag{3.30}$$

The third term, according to (3.25) and (3.30), is finally bounded by

$$C_{k,3}\omega^{k-1}\varphi(\varepsilon)^{k-1/2}\,\|v\|_{\mathrm{H}^1(\widehat{\Omega}_\alpha)} \quad \text{with } C_{k,3} \text{ which does not depend or } \varepsilon \text{ nor } \omega \tag{3.31}$$

The first and the second terms can be treated in the same way. The idea is to do the same thing as in the section 2.3.2, the only difference is that our sums are finite sums, instead of being infinite sums. We can see that if we take $\varphi_\varepsilon = (2k)\varepsilon|\ln(\varepsilon)|$, the terms $d^m_{p,\pm}$ are $o(\varepsilon^k)$ up to $m = k$, for ε small enough. Then, it is quite easy to show that the first term is bounded by

$$C_{k,1}\ \varepsilon^k \varphi(\varepsilon)^{-2} \omega^{k+1} \int_0^{2\varphi(\varepsilon)} \int_{-\varepsilon}^{0} \overline{v}(\pm s, \nu) d\nu ds \tag{3.32}$$

and the second term is bounded by

$$C_{k,2}\ \varepsilon^{k-1} \varphi(\varepsilon)^{-1} \omega^{k} \int_0^{2\varphi(\varepsilon)} \int_{-\varepsilon}^{0} \overline{v}(\pm s, \nu) d\nu ds \tag{3.33}$$

with constants $C_{k,1}$ and $C_{k,2}$ that do not depend on ε nor ω.
We use again the lemma (3.7) and we use that $\varepsilon \leqslant \varphi(\varepsilon)$, since $\varphi(\varepsilon)\varepsilon^{-1}$ tends to ∞ as ε tends to 0, then we can say that there exist two constants we call again $C_{k,1}$ and $C_{k,2}$ which do not depend on ε nor ω such that the two first terms of (3.23) are bounded by

$$\left(C_{k,1}\omega^{k+1} + C_{k,2}\omega^k\right) \varphi(\varepsilon)^{k-1/2} \|v\|_{\mathrm{H}^1(\widehat{\Omega}_\alpha)} \tag{3.34}$$

Finally, using (3.31) and (3.34) in (3.23) gives , taking $v = u^\varepsilon - u^{\varepsilon,k}_{app}$ and denoting $C_k = C_{k,1}\omega^{k+1} + C_{k,2}\omega^k + C_{k,3}\omega^{k-1}$

$$|a^\varepsilon(u^\varepsilon - u^{\varepsilon,k}_{app}, u^\varepsilon - u^{\varepsilon,k}_{app})| \ \leqslant\ C_k \varphi(\varepsilon)^{k-1/2} \left\|u^\varepsilon - u^{\varepsilon,k}_{app}\right\|_{\mathrm{H}^1(\widehat{\Omega}_\alpha)} \tag{3.35}$$

To conclude, we use the stability result given by the proposition B.1, and we use $\varphi(\varepsilon) = (2k)\varepsilon|\ln(\varepsilon)|$, to have the following theorem

Theorem 3.8. *There exists a constant $\tilde{C}_k$ which does not depend on ε such that, for ε small enough,*

$$\left\|u^\varepsilon - u^{\varepsilon,k}_{app}\right\|_{\mathrm{H}^1(\widehat{\Omega}_\alpha)} \leqslant \tilde{C}_k \left(\varepsilon|\ln(\varepsilon)|\right)^{k-1/2} \tag{3.36}$$

Remark 3.9. In this theorem, the constant $\tilde{C}_k$ is like ω^k. This means, if we go back to the time-domain equation, that the more precise approximate function we want, the more derivates of the Cauchy data with respect to time we have to consider.

3.2.2 Local error estimate

We can get a better result if we only look the error in the slots $\widehat{\Omega}^\delta_\pm$ (see back 1.19). We can write that

$$\widehat{u}^\varepsilon - \widehat{u}^{\varepsilon,k}_{app} = \left(\widehat{u}^\varepsilon - \widehat{u}^{\varepsilon,k+2}_{app}\right) + \left(\widehat{u}^{\varepsilon,k+2}_{app} - \widehat{u}^{\varepsilon,k}_{app}\right) \tag{3.37}$$

and some classical triangular inegality gives

$$\left\|\widehat{u}^\varepsilon - \widehat{u}^{\varepsilon,k}_{app}\right\|_{\mathrm{H}^1(\widehat{\Omega}^\delta_\pm)} \leqslant \left\|\widehat{u}^\varepsilon - \widehat{u}^{\varepsilon,k+2}_{app}\right\|_{\mathrm{H}^1(\widehat{\Omega}^\delta_\pm)} + \left\|\widehat{u}^{\varepsilon,k+2}_{app} - \widehat{u}^{\varepsilon,k}_{app}\right\|_{\mathrm{H}^1(\widehat{\Omega}^\delta_\pm)} \tag{3.38}$$

Thanks to the theorem 3.8, we can raise up the term $\left\|\widehat{u}^\varepsilon - \widehat{u}_{app}^{\varepsilon,k+2}\right\|_{\mathrm{H}^1(\widehat{\Omega}_\pm^\delta)}$ by $C'_k\ \varepsilon^{k+1}$ when ε is small enough (in fact, one can see that we can always do this majoration, since ε is small enough thanks to the hypothesis of this theorem). For the second term, we can see that for ε such that $4k\varepsilon|\ln(\varepsilon)| \leqslant \delta$ (in other words, $2\varphi(\varepsilon) \leqslant \delta$), we have, on $\widehat{\Omega}_\pm^\delta$:

$$\widehat{u}_{app}^{\varepsilon,k+2} - \widehat{u}_{app}^{\varepsilon,k} = (\varepsilon\omega)^{k+1} u_\pm^{k+1} + (\varepsilon\omega)^{k+2} u_\pm^{k+2} \tag{3.39}$$

and we can easily see, since the family of functions $u_\pm^k$ does not depend on ε, that the norm of (3.39) is bounded by a constant C_δ times ε^{k+1}.
Moreover, we can express the following theorem

Theorem 3.10. *For any $0 < \delta < \delta^*$, we can build up the 1D approximates functions*

$$\tilde{u}^{\varepsilon,k}(s,\nu) = \sum_{m=0}^{k} (\varepsilon\omega)^m u_\pm^m, \quad \textit{for } \pm s \geqslant \delta \textit{ and for } -\varepsilon < \nu < 0 \tag{3.40}$$

Then :

1. *for ε such that $4k\varepsilon|\ln(\varepsilon)| \leqslant \delta$, we have $\tilde{u}^{\varepsilon,k} = u_{app}^{\varepsilon,k}$ on $\widehat{\Omega}_\pm^\delta$,*
2. *we have the following error estimate : there exists a constant C_δ which does not depend on ε such that*
$$\left\|u^\varepsilon - \tilde{u}^{\varepsilon,k}\right\|_{\mathrm{H}^1(\widehat{\Omega}_\pm^\delta)} \leqslant C_\delta\ \varepsilon^{k+1} \tag{3.41}$$

Proof. The proof of this theorem is straightforward. ■

One good remark we can do is : what happens if we take directly $\delta = 0$ in the theorem 3.10? In fact, we have the following result

Proposition 3.11. *We have the following equivalent : for any $k \in \mathbb{N}$,*

$$\left\|u^\varepsilon - \tilde{u}^{\varepsilon,k}\right\|_{\mathrm{H}^1(\widehat{\Omega}_\pm)} \sim C\varepsilon \tag{3.42}$$

Proof. Near $\pm s = 0^+$, we can see that the function $\tilde{u}^{\varepsilon,k}$ is clearly a $1D$ function, however u^ε is a real $2D$ function. By using some Fourier analysis, and if we call $c_\pm^p(u^\varepsilon)$ the $1D$ function defined by

$$c_\pm^p(u^\varepsilon)(s) = \int_{-1}^{0} u^\varepsilon(s,\varepsilon\widehat{\nu}) w_p(\widehat{\nu}) d\widehat{\nu}, \quad \pm s > 0 \tag{3.43}$$

we can easily see that

$$\left\|u^\varepsilon - \tilde{u}^{\varepsilon,k}\right\|_{\mathrm{H}^1(\widehat{\Omega}_\pm)} = \left\|c_0(u^\varepsilon) - \tilde{u}^{\varepsilon,k}\right\|_{\mathrm{H}^1((0,\pm L\pm))} + \sum_{p\in\mathbb{N}^*} (1+p^2) \left\|c_p(u^\varepsilon)\right\|_{\mathrm{H}^1((0,\pm L\pm))} \tag{3.44}$$

However, by using the modal expansion of the functions U^k, it is quite easy to show that $\|c_p(u^\varepsilon)\|_{\mathrm{H}^1((0,\pm L\pm))} \sim \varepsilon$ (because U^1 is a real $2D$ function), and that explains (3.42). ■

Remark 3.12. Taking $k = 0$ in the proposition 3.11 added to the remark 3.6 let us allow to write down the result of the theorem 1.3 (we knew already the limit fonction, the new fact here is that we have the error estimates).

4 Construction and analysis of the improved Kirchoff conditions

In this section, once we proved by the theorem 3.5 that the functions $u^k_\pm$ and the functions U^k are uniquely defined, we explain how we can get the improved 1D problem (1.16, 1.17) and how we can get the error (1.20) of the theorem 1.5.

4.1 Construction of the Kirchoff conditions

To improve the transmission conditions satisfied by the limit problem, according to theorem 3.10, it is natural to look at the transmission conditions satisfied by:

$$u^{\varepsilon,1} = u^0 + \varepsilon\omega\, u^1$$

For the Neumann jump condition, we first use (3.14) for k = 1, the fact that $U^0 = u^0(0)$ in $\widehat{J}_\alpha$ and that the measure of $\widehat{J}_\alpha$ is $\tan\alpha$ to get

$$\left[\frac{\partial u^1}{\partial s}\right] = -\omega \int_{\widehat{J}_\alpha} U^0 \; = -\omega\; \tan\alpha\, u^0(0) \equiv -\omega\; \tan\alpha\; \left\langle u^0 \right\rangle \;. \tag{4.1}$$

Thus $u^{\varepsilon,1}$ satisfies

$$\left[\frac{\partial u^{\varepsilon,1}}{\partial s}\right] = \left[\frac{\partial u^0}{\partial s}\right] + \varepsilon\omega\; \left[\frac{\partial u^1}{\partial s}\right] = -\varepsilon\,\omega^2\; \tan\alpha\; \left\langle u^0 \right\rangle$$

which can be rewritten, since $u^0 = u^{\varepsilon,1} - \varepsilon\omega\, u^1$

$$\left[\frac{\partial u^{\varepsilon,1}}{\partial s}\right] + \varepsilon\,\omega^2\; \tan\alpha\; \left\langle u^{\varepsilon,1} \right\rangle = O(\varepsilon^2). \tag{4.2}$$

For the Dirichlet jump condition, we first use (3.16) for k = 1,

$$\left[u^1\right] = \int_{\Sigma_-^\alpha} g_-^0\; \mathcal{W}_\alpha + \int_{\Sigma_+^\alpha} g_+^0\; \mathcal{W}_\alpha \;.$$

with, using (3.13) for $k = 1$,

$$g^0_\pm := \pm\frac{1}{\omega}\frac{\partial u^0_\pm}{\partial s}(0) \equiv \pm\;\frac{1}{\omega}\left\langle \frac{\partial u^0}{\partial s}\right\rangle$$

Thus, by definition of $K(\alpha)$,

$$\left[u^1\right] = \frac{1}{\omega}\; K(\alpha)\; \left\langle \frac{\partial u^0}{\partial s}\right\rangle. \tag{4.3}$$

Therefore, $u^{\varepsilon,1}$ satisfies

$$\left[u^{\varepsilon,1}\right] = \left[u^0\right] + \varepsilon\omega \ \left[u^1\right] = \varepsilon \ K(\alpha) \ \left\langle \frac{\partial u^0}{\partial s} \right\rangle$$

which can be rewritten, using again $u^0 = u^{\varepsilon,1} - \varepsilon\omega \ u^1$

$$\left[u^{\varepsilon,1}\right] - \varepsilon \ K(\alpha) \ \left\langle \frac{\partial u^{\varepsilon,1}}{\partial s} \right\rangle = O(\varepsilon^2). \tag{4.4}$$

Finally, the transmission conditions for the 1D approximate solution $\tilde{u}^\varepsilon$, namely (1.17) , are simply obtained by dropping the $O(\varepsilon^2)$ terms in (4.4) and (4.2) repectively.

4.2 Analysis of the transmission conditions

To prove the result (1.20), we will make an asymptotic expansion on (1.16, 1.17) and then we will compare the asymptotic expansion got here with the asymptotic expansion of the theorem 3.5. The precise meaning is the following :

(i) For the problem (1.16, 1.17), we write $\tilde{u}^\varepsilon$ as a power serie of ε, in fact

$$\tilde{u}^\varepsilon = \sum_{k\in\mathbb{N}} (\varepsilon\omega)^n \tilde{u}^k \tag{4.5}$$

We can easily see that each function $\tilde{u}^k$ satisfies the Helmholtz equation for $\pm s > 0$ with boundary conditions the same boundary conditions as u^k at $s = \pm L_\pm$, and each function $\tilde{u}^k$ satisfies the following transmission conditions

$$\left[\tilde{u}^k\right] = \frac{1}{\omega} K(\alpha) \left\langle \frac{\partial \tilde{u}^{k-1}}{\partial s} \right\rangle \quad \text{and} \quad \left[\frac{\partial \tilde{u}^k}{\partial s}\right] = -\omega \tan(\alpha) \left\langle \tilde{u}^{k-1} \right\rangle \tag{4.6}$$

Note that the family $(\tilde{u})_{k\in\mathbb{N}}$ is uniquely defined thanks to the lemma 3.3.

(ii) One has the following theorem

Theorem 4.1. *There exists a constant C_k independant of ε such that*

$$\left\| \tilde{u}^\varepsilon - \sum_{m=0}^{k} (\varepsilon\omega)^m \tilde{u}^m \right\|_{\mathrm{H}^1((0,\pm L_\pm))} \leqslant C_k \varepsilon^{k+1} \tag{4.7}$$

This theorem will be proved (for the clarty of this section) in the appendix C.

(iii) One observes that $\widetilde{u}^0 = u^0$, $\widetilde{u}^1 = u^1$ and $\widetilde{u}^2 = u^2$, the last equality being an unexpected bonus linked to the fact that $\widehat{J}_\alpha$ has an symmetry axis. Indeed, (3.6) for $k = 1$ gives

$$\frac{\partial U^1}{\partial n} + T_\pm U^1 = \pm\frac{1}{\omega}\frac{\partial u^0_\pm}{\partial s}(0) \equiv \pm\frac{1}{\omega}\left\langle \frac{\partial u^0}{\partial s} \right\rangle.$$

So, using the definition of $\mathcal{W}_\alpha$ (cf. (1.11, (1.10)) and (3.17) for $k = 1$, we conclude via lemma 3.2 that

$$U^1 = \langle u^1 \rangle + \frac{1}{\omega} \left\langle \frac{\partial u^0}{\partial s} \right\rangle (0)\, \mathcal{W}_\alpha \ .$$

Thus, writing (3.14) for k = 2 we get

$$\left[\frac{\partial u^2}{\partial s} \right] = -\omega \int_{\widehat{J}_\alpha} U^1 \ = -\omega \ \tan\alpha \ \langle u^1 \rangle \ . \tag{4.8}$$

which is identical to (4.1) after index shifting. On the other hand, writing (3.16) for k = 2 gives

$$\left[u^2 \right] = \int_{\Sigma_-^\alpha} g_-^1 \ \mathcal{W}_\alpha + \int_{\Sigma_+^\alpha} g_+^1 \ \mathcal{W}_\alpha \ .$$

where, using (3.13) for $k = 1$ and $\delta_p^1 = 0$, $g_\pm^1 := \pm \dfrac{1}{\omega} \dfrac{\partial u_\pm^1}{\partial s}(0)$. Thus

$$\left[u^2 \right] = \frac{1}{\omega} \ \left(\frac{\partial u_+^1}{\partial s}(0) \int_{\Sigma_+^\alpha} \mathcal{W}_\alpha - \frac{\partial u_-^1}{\partial s}(0) \int_{\Sigma_-^\alpha} \mathcal{W}_\alpha \right).$$

Since $\widehat{J}_\alpha$ has an axis of symmetry, $\mathcal{W}_\alpha$ is anti-symmetric with respect to this line (see more precisely the proposition 5.1) and consequently

$$\int_{\Sigma_+^\alpha} \mathcal{W}_\alpha = - \int_{\Sigma_-^\alpha} \mathcal{W}_\alpha = \frac{K(\alpha)}{2}. \quad \text{(using the definition (1.15) of } K(\alpha) \text{)}$$

Finally, we have

$$\left[u^2 \right] = \frac{1}{\omega} \ K(\alpha) \ \left\langle \frac{\partial u^1}{\partial s} \right\rangle . \tag{4.9}$$

which is nothing but (4.3) after index shifting.

(iv) This allows us to write in $\widehat{\Omega}_\pm^\delta$ (modulo some abuse of notation)

$$u^\varepsilon - \tilde{u}^\varepsilon = \left(u^\varepsilon - \tilde{u}^{\varepsilon,2} \right) + \left(\sum_{m=0}^{2} (\varepsilon\omega)^m \tilde{u}^m - \tilde{u}^\varepsilon \right)$$

and one concludes using (4.7) of the theorem 4.1 for $k = 2$, (3.41) of the theorem 3.10 for $k = 2$ and the triangular inequality to get (1.20).

5 Study of the reflexion, transmission waves

We consider here the case of the junction of two semi-infinite strips, i.e. $L_\pm = +\infty$ and look at the approximate model, i.e. the 1D Hemlholtz equation for $\pm s > 0$ with the transmission conditions (1.17) at $s = 0$. We are more precisely interested in the reflection-transmission of an incident wave in the domain $s < 0$, i. e. look for a solution of the form

$$u(s) = \begin{cases} \exp(\imath\omega s) + R_\varepsilon(\alpha,\omega)\ \exp(-\imath\omega s),\ s < 0 \\ T_\varepsilon(\alpha,\omega)\ \exp(\imath\omega s),\ s > 0 \end{cases} \tag{5.1}$$

If we put the expression of the total field (5.1) in the improved Kirchhoff laws we got in (1.17), we get

$$\left\{ \begin{array}{rcl} T_\varepsilon(\alpha,\omega) + R_\varepsilon(\alpha,\omega) - 1 & = & \dfrac{\imath\varepsilon\omega\tan(\alpha)}{2}(T_\varepsilon(\alpha,\omega) + R_\varepsilon(\alpha,\omega) + 1) \\ \\ T_\varepsilon(\alpha,\omega) - R_\varepsilon(\alpha,\omega) - 1 & = & \dfrac{\imath\varepsilon\omega K(\alpha)}{2}(T_\varepsilon(\alpha,\omega) - R_\varepsilon(\alpha,\omega) + 1) \end{array} \right. \tag{5.2}$$

The equations (5.2) gives then that

$$T_\varepsilon(\alpha,\omega) = \frac{1}{2}\left(\frac{1 + \frac{\imath\varepsilon\omega\tan(\alpha)}{2}}{1 - \frac{\imath\varepsilon\omega\tan(\alpha)}{2}} + \frac{1 + \frac{\imath\varepsilon\omega K(\alpha)}{2}}{1 - \frac{\imath\varepsilon\omega K(\alpha)}{2}} \right) \tag{5.3}$$

$$R_\varepsilon(\alpha,\omega) = \frac{1}{2}\left(\frac{1 + \frac{\imath\varepsilon\omega\tan(\alpha)}{2}}{1 - \frac{\imath\varepsilon\omega\tan(\alpha)}{2}} - \frac{1 + \frac{\imath\varepsilon\omega K(\alpha)}{2}}{1 - \frac{\imath\varepsilon\omega K(\alpha)}{2}} \right) \tag{5.4}$$

The figures 5.1 to 5.4 show the modulus of the coefficients $T_\varepsilon(\alpha,\omega)$ and $R_\varepsilon(\alpha,\omega)$ with respect to α, with different values of $\varepsilon\omega$. By looking for these results, and by looking for the expressions of $T_\varepsilon(\alpha,\omega)$ and $R_\varepsilon(\alpha,\omega)$ given by (5.3) and (5.4), we can say that :

- For a given $\alpha \in [0,\pi/2[$, and for $\varepsilon\omega\tan(\alpha)$ small enough (that implies $\varepsilon\omega$ is small enough), we can write the coefficients $T_\varepsilon(\alpha,\omega)$ and $R_\varepsilon(\alpha,\omega)$ as

$$T_\varepsilon(\alpha,\omega) = 1 + \frac{\imath\varepsilon\omega}{2}(\tan(\alpha) + K(\alpha)) + O((\varepsilon\omega)^2) \tag{5.5}$$

$$R_\varepsilon(\alpha,\omega) = \frac{\imath\varepsilon\omega}{2}(\tan(\alpha) - K(\alpha)) + O((\varepsilon\omega)^2) \tag{5.6}$$

- A consequence of (5.5) and (5.6) is that

$$|R_\varepsilon(\alpha,\omega)| = O(\varepsilon), \quad |T_\varepsilon(\alpha,\omega)| = 1 + O(\varepsilon^2). \tag{5.7}$$

  which means that in practice, the reflection phenomenon is in amplitude more directly visible than the transmission phenomenon.

Figures 5.1 to 5.4 : plot of $|T_\varepsilon(\alpha,\omega)|$ and $|R_\varepsilon(\alpha,\omega)|$ with respect to α in degrees

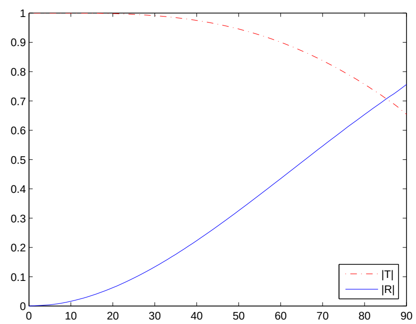

Figure 5.1: $\varepsilon\omega = 1$

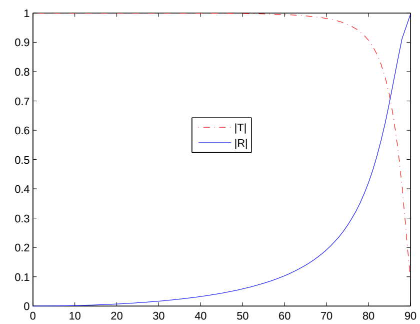

Figure 5.2: $\varepsilon\omega = 0.1$

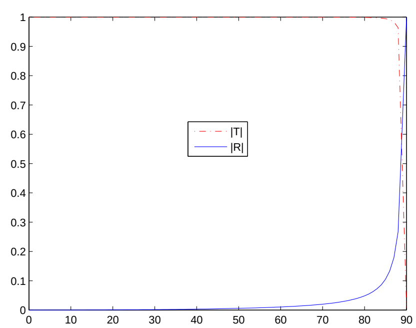

Figure 5.3: $\varepsilon\omega = 0.01$

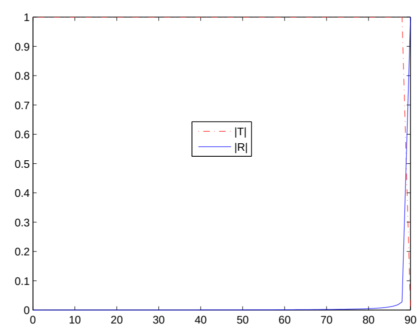

Figure 5.4: $\varepsilon\omega = 0.001$

- For a given $\varepsilon\omega$, and for α tending to zero, the expressions (5.5) and (5.6) are still valid, except that the "$O((\varepsilon\omega)^2)$" term becomes a "$O(\alpha^2)$" term.
- For a given $\varepsilon\omega$, and for α tending to $\pi/2$, the expressions (5.3) and (5.4) become

$$T_\varepsilon(\alpha,\omega) = \frac{1}{2}\left(-1 + \frac{1 + \frac{\imath\varepsilon\omega K(\pi/2)}{2}}{1 - \frac{\imath\varepsilon\omega K(\pi/2)}{2}}\right) \tag{5.8}$$

$$R_\varepsilon(\alpha,\omega) = \frac{1}{2}\left(-1 - \frac{1 + \frac{\imath\varepsilon\omega K(\pi/2)}{2}}{1 - \frac{\imath\varepsilon\omega K(\pi/2)}{2}}\right) \tag{5.9}$$

Here, in the expressions (5.8) and (5.9), we can see that when we let $\varepsilon\omega$ tend to zero, $T_\varepsilon(\alpha,\omega)$ tends to 0 and $R_\varepsilon(\alpha,\omega)$ tends to -1. We can see moreover that

$$\lim_{\alpha\to\frac{\pi}{2}}\lim_{\varepsilon\omega\to 0}|R_\varepsilon(\alpha,\omega)| = 0 \qquad \text{and} \qquad \lim_{\varepsilon\omega\to 0}\lim_{\alpha\to\frac{\pi}{2}}|R_\varepsilon(\alpha,\omega)| = 1$$

By looking for expression of the coefficient R given by (5.6), we can ask ourself : how does this coefficient alter with respect to α ? To answer, we need some additional informations about the coefficient $K(\alpha)$

5.1 More informations about the coefficient $K(\alpha)$

We recall that $K(\alpha)$ is given by the function $\mathcal{W}_\alpha$ solution of the problem (1.11, 1.10). We can even reduce the computation domain thanks to the following proposition

Proposition 5.1. *On the axis* $\Sigma_0^\alpha := \{(-Y\tan(\alpha),\ Y)$ *with* $Y \in]-1,0[\}$, *the function* $\mathcal{W}_\alpha$ *vanishes, and* $K(\alpha)$ *can instead be given*

$$K(\alpha) = 2\int_{\Sigma_+^\alpha}\mathcal{W}_\alpha \tag{5.10}$$

Proof. We can note that $\widehat{J}_\alpha$ is symmetric with respect to the axis Σ_0^α. On this domain, we introduced the "rotated" function

$$\tilde{\mathcal{W}}_\alpha(\tilde{\mathbf{x}}) = \mathcal{W}_\alpha(\mathbf{x}), \quad \text{for } \tilde{\mathbf{x}} = (\mathbf{x}\cdot(\cos(\alpha),\sin(\alpha)), \mathbf{x}\cdot(-\sin(\alpha),\cos(\alpha))) \tag{5.11}$$

and we introduce the rotated set $\tilde{J}_\alpha$ by the set

$$\tilde{J}_\alpha = \left\{\tilde{\mathbf{x}} \in \mathbb{R}^2 \text{ such as } \tilde{\mathbf{x}} = (\mathbf{x}\cdot(\cos(\alpha),\sin(\alpha)), \mathbf{x}\cdot(-\sin(\alpha),\cos(\alpha))) \text{ with } \mathbf{x}\in\widehat{J}_\alpha\right\} \tag{5.12}$$

We can see that proving the proposition 5.1 is strictly equivalent to prove that

$$\tilde{\mathcal{W}}_\alpha(\tilde{\mathbf{x}}) = 0 \text{ for } (\tilde{\mathbf{x}}) \ = \ (0,\tilde{y})$$

Since $\tilde{J}_\alpha$ is symmetric with respect to the axis $\tilde{x}=0$, we just have to prove that $\tilde{\mathcal{W}}_\alpha(\cdot,\tilde{y})$ is odd. We spilt $\tilde{\mathcal{W}}_\alpha$ into its even part $\tilde{\mathcal{W}}_\alpha^e$ and its odd part $\tilde{\mathcal{W}}_\alpha^o$ (with respect to $\tilde{x}$ of course). Since the odd part has a mean-value 0 and thanls to (1.10), we have

$$\int_{\tilde{J}_\alpha}\tilde{\mathcal{W}}_\alpha^e = 0 \tag{5.13}$$

and always by spilting the odd and the even part, we can see that

$$\begin{cases} \Delta\tilde{\mathcal{W}}_\alpha^e = 0, & \text{in } \tilde{J}_\alpha, \\[2mm] \dfrac{\partial\tilde{\mathcal{W}}_\alpha^e}{\partial n} + T_\pm\tilde{\mathcal{W}}_\alpha^e = 0, & \text{on } \tilde{\Sigma}_\pm^\alpha, \\[2mm] \dfrac{\partial\tilde{\mathcal{W}}_\alpha^e}{\partial n} = 0, & \text{on } \partial\tilde{J}_\alpha \setminus \left(\tilde{\Sigma}_+^\alpha \cup \tilde{\Sigma}_-^\alpha\right) \end{cases} \tag{5.14}$$

(5.14, 5.13) give that $\tilde{\mathcal{W}}_\alpha^e = 0$, and the proof is complete (since the function $\tilde{\mathcal{W}}_\alpha$ is an odd function, the mean-value over the bound Σ_-^α is the opposite of the mean-value over the bound Σ_+^α). ■

Once we proved this proposition, we are interested by the set $\mathbf{J}_\alpha$ defined by (see the figure 5.5)

$$\mathbf{J}_\alpha = \left\{ \widehat{\mathbf{x}} \in \widehat{J}_\alpha \text{ such as } \widehat{\mathbf{x}} \cdot (\cos(\alpha),\ \sin(\alpha)) > 0 \right\} \tag{5.15}$$

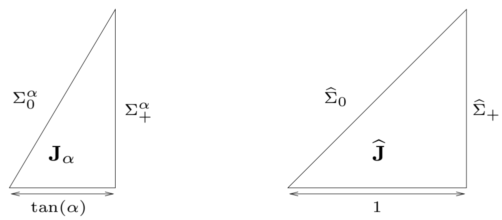

Figure 5.5: Configuration of the domain $\mathbf{J}_\alpha$ and the canonical domain $\widehat{\mathbf{J}}$

We rotate the set $\mathbf{J}_\alpha$ to put the set Σ_+^α onto the set $\{0\}\times]-1,0[$. We see that $\mathcal{W}_\alpha$ is solution of the following problem (in the new coordinates, and thanks to the proposition 5.1) :

$$\left\{ \begin{array}{ll} \Delta \mathcal{W}_\alpha = 0, & \text{in } \mathbf{J}_\alpha, \\[2mm] \dfrac{\partial \mathcal{W}_\alpha}{\partial X} + T_+ \mathcal{W}_\alpha = 1, & \text{on } \Sigma_+^\alpha \\[2mm] \mathcal{W}_\alpha = 0, & \text{on } \Sigma_0^\alpha \\[2mm] \dfrac{\partial \mathcal{W}_\alpha}{\partial Y} = 0, & \text{on the remaining boundary} \end{array} \right. \tag{5.16}$$

and we get the coefficient $K(\alpha)$ by the relation (5.10). However, it is more judicious to introduce the $1D$ scaled function

$$\Phi_\alpha(\tilde{X}, \tilde{Y}) = \frac{1}{\tan(\alpha)} \mathcal{W}_\alpha(\tan(\alpha)\tilde{X}, \tilde{Y}), \text{ for } (\tilde{X}, \tilde{Y}) \in \widehat{\mathbf{J}} \tag{5.17}$$

and we see that Φ_α satisfies the following problem

$$\begin{cases} \dfrac{\partial^2 \Phi_\alpha}{\partial \tilde{X}^2} + \tan^2(\alpha) \dfrac{\partial^2 \Phi_\alpha}{\partial \tilde{Y}^2} = 0, & \text{in } \widehat{\mathbf{J}}, \\ \dfrac{\partial \Phi_\alpha}{\partial \tilde{X}} + \tan(\alpha) T_+ \Phi_\alpha = 1, & \text{on } \widehat{\Sigma}_+, \\ \Phi_\alpha = 0, & \text{on } \widehat{\Sigma}_0, \\ \dfrac{\partial \Phi_\alpha}{\partial \tilde{Y}} = 0, & \text{on the remaining boundary } \widehat{\Sigma}_Y. \end{cases} \tag{5.18}$$

Reciproquely, if Φ_α is a solution of (5.18), then the restriction of $\mathcal{W}_\alpha$ to the semi-junction $\mathbf{J}_\alpha$ is a solution of (5.16). The interest of the function Φ_α is that this function is defined on a domain that do not depend on the value of α.
We can see that it is natural to introduce the following space

$$\mathrm{H}^1_0(\widehat{\mathbf{J}}) = \left\{ U \in \mathrm{H}^1(\widehat{\mathbf{J}}) \text{ such as } U = 0 \text{ on } \widehat{\Sigma}_0 \right\} \tag{5.19}$$

Since we search a function in $\mathrm{H}^1_0(\widehat{\mathbf{J}})$ satisfying the problem (5.18), it is natural to associate the variationnal problem : find $\Phi_\alpha \in \mathrm{H}^1_0(\widehat{\mathbf{J}})$ such that for all $\mathcal{V} \in \mathrm{H}^1_0(\widehat{\mathbf{J}})$,

$$\int_{\widehat{\mathbf{J}}} \left(\frac{\partial \Phi_\alpha}{\partial \tilde{X}} \frac{\partial \mathcal{V}}{\partial \tilde{X}} + \tan^2(\alpha) \frac{\partial \Phi_\alpha}{\partial \tilde{Y}} \frac{\partial \mathcal{V}}{\partial \tilde{Y}} \right) + \tan(\alpha) \int_{\widehat{\Sigma}_+} \mathcal{V} T_+ \Phi_\alpha = \int_{\widehat{\Sigma}_+} \mathcal{V} \tag{5.20}$$

Then, we get the following trace on $\widehat{\Sigma}_+$:

$$\int_{\widehat{\Sigma}_+} \Phi_\alpha = \frac{K(\alpha)}{2 \tan(\alpha)} \tag{5.21}$$

Proposition 5.2. *For a given $\alpha \in]0, \pi/2[$, the problem (5.20) has a unique solution.*

Proof. It is simply the use of the Lax-Milgram theorem, as for the proof od the proposition 1.4. ∎

Here, we will give three properties about the function $\alpha \mapsto \frac{K(\alpha)}{\tan(\alpha)}$:

Proposition 5.3. *The function $\alpha \mapsto \frac{K(\alpha)}{\tan(\alpha)}$ is a decreasing function with respect to α*

Proof. Because of the writing of the trace (5.21), it is quite natural to introduce Ψ_α as the derivate of Φ_α with respect to α. Let is derive the problem (5.18) with respect to α : we get

$$\left\{\begin{array}{ll} \dfrac{\partial^2\Psi_\alpha}{\partial\tilde{X}^2} + \tan^2(\alpha)\dfrac{\partial^2\Psi_\alpha}{\partial\tilde{Y}^2} + 2\tan(\alpha)(1+\tan^2(\alpha))\dfrac{\partial^2\Psi_\alpha}{\partial\tilde{Y}^2} = 0, & \text{in } \widehat{\mathbf{J}}, \\ \dfrac{\partial\Psi_\alpha}{\partial\tilde{X}} + \tan(\alpha)T_+\Psi_\alpha + (1+\tan^2(\alpha))\Phi_\alpha = 0, & \text{on } \widehat{\Sigma}_+, \\ \Psi_\alpha = 0, & \text{on } \widehat{\Sigma}_0, \\ \dfrac{\partial\Psi_\alpha}{\partial\tilde{Y}} = 0, & \text{on } \widehat{\Sigma}_Y \end{array}\right. \tag{5.22}$$

And we associate the variationnal problem : find $\Psi_\alpha \in \mathrm{H}^1_0(\widehat{\mathbf{J}})$ such that for all $\mathcal{V} \in \mathrm{H}^1_0(\widehat{\mathbf{J}})$,

$$\begin{aligned} \int_{\widehat{\mathbf{J}}} \left(\frac{\partial\Psi_\alpha}{\partial\tilde{X}}\frac{\partial\mathcal{V}}{\partial\tilde{X}} + \tan^2(\alpha)\frac{\partial\Psi_\alpha}{\partial\tilde{Y}}\frac{\partial\mathcal{V}}{\partial\tilde{Y}} + 2\tan(\alpha)(1+\tan^2(\alpha))\frac{\partial\Phi_\alpha}{\partial\tilde{Y}}\frac{\partial\mathcal{V}}{\partial\tilde{Y}}\right) \\ + \tan(\alpha)\int_{\widehat{\Sigma}_+} \mathcal{V}T_+\Psi_\alpha + (1+\tan^2\alpha)\int_{\widehat{\Sigma}_+} \mathcal{V}T_+\Phi_\alpha = 0 \end{aligned} \tag{5.23}$$

The idea, then, is to use (5.20) with the particular test function $\mathcal{W} = \Psi_\alpha$, and to use (5.23) with the particular test function $\mathcal{W} = \Phi_\alpha$. We get that (using the fact that $\displaystyle\int_{\widehat{\Sigma}_+} \Phi_\alpha T_+\Psi_\alpha = \int_{\widehat{\Sigma}_+} \Psi_\alpha T_+\Phi_\alpha$) :

$$\int_{\widehat{\Sigma}_+} \Psi_\alpha = -2\tan(\alpha)(1+\tan^2(\alpha))\left\|\frac{\partial\Phi_\alpha}{\partial\tilde{Y}}\right\|^2_{\mathrm{L}^2(\widehat{\mathbf{J}})} - (1+\tan^2(\alpha))\int_{\widehat{\Sigma}_+} \Phi_\alpha T_+\Phi_\alpha \tag{5.24}$$

Since the trace of Ψ_α over $\widehat{\Sigma}_+$ is the derivate of the function we want to study, and thanks to the proposition A.5, the proof is complete. ∎

Proposition 5.4. *When $\alpha \to 0$, the function Φ_α converges (at least in a L^2 sense) to the function $\Phi_0 : (\tilde{X},\tilde{Y}) \mapsto \tilde{X} - \tilde{Y}$, and we have $K(\alpha) \sim \tan(\alpha)$*

Proof. We start from the initial problem (5.20). Formally, if we let α tends to zero, we can see that the limit problem obtained is : find $\Phi_0 \in \mathrm{H}^1_0(\widehat{\mathbf{J}})$ such that for all $\mathcal{V} \in \mathrm{H}^1_0(\widehat{\mathbf{J}})$,

$$\int_{\widehat{\mathbf{J}}} \frac{\partial\Phi_0}{\partial\tilde{X}}\frac{\partial\mathcal{V}}{\partial\tilde{X}} = \int_{\widehat{\Sigma}_+} \mathcal{V} \tag{5.25}$$

The associated problem is :

$$\left\{\begin{array}{ll} \dfrac{\partial^2\Phi_0}{\partial\tilde{X}^2} = 0, & \text{in } \widehat{\mathbf{J}}, \\ \dfrac{\partial\Phi_0}{\partial\tilde{X}} = 1, & \text{on } \widehat{\Sigma}_+, \\ \Phi_0 = 0, & \text{on } \widehat{\Sigma}_0. \end{array}\right. \tag{5.26}$$

This problem has a unique solution, which is $\Phi_0 : (\tilde{X},\tilde{Y}) \mapsto \tilde{X} - \tilde{Y}$. Then, we can see that the trace of Φ_0 over $\widehat{\Sigma}_+$ is equal to $\frac{1}{2}$.
To prove that Φ_α converges to Φ_0, we make the difference between the two variationnal formulations (5.20) and (5.25), to get

$$\begin{aligned} \int_{\widehat{\mathbf{J}}} \left(\frac{\partial(\Phi_\alpha - \Phi_0)}{\partial\tilde{X}} \frac{\partial\mathcal{V}}{\partial\tilde{X}} + \tan^2(\alpha) \frac{\partial(\Phi_\alpha - \Phi_0)}{\partial\tilde{Y}} \frac{\partial\mathcal{V}}{\partial\tilde{Y}} \right) + \tan(\alpha) \int_{\widehat{\Sigma}_+} \mathcal{V} T_+(\Phi_\alpha - \Phi_0) \\ = -\tan^2(\alpha) \int_{\widehat{\mathbf{J}}} \frac{\partial\Phi_0}{\partial\tilde{Y}} \frac{\partial\mathcal{V}}{\partial\tilde{Y}} - \tan(\alpha) \int_{\widehat{\Sigma}_+} \mathcal{V} T_+ \Phi_0 \end{aligned} \tag{5.27}$$

We look for the problem (5.27) with the "weighted" norm

$$\|U\|^2_{\mathrm{H}_\alpha(\widehat{\mathbf{J}})} = \left\| \frac{\partial U}{\partial\tilde{X}} \right\|^2_{\mathrm{L}^2(\widehat{\mathbf{J}})} + \tan^2(\alpha) \left\| \frac{\partial U}{\partial\tilde{Y}} \right\|^2_{\mathrm{L}^2(\widehat{\mathbf{J}})} + \tan(\alpha) \int_{\widehat{\Sigma}_+} U T_+ U \tag{5.28}$$

Note that, for a given $\alpha \in]0, \pi/2[$, the norm defined in (5.28) is equivalent to the classical H^1 seminorm on $\widehat{\mathbf{J}}$ (which is equivalent here to classical norm thanks to the Poincaré inegality). After taking $\mathcal{W} = \Phi_\alpha - \Phi_0$ in (5.27), we can see that the left member is the square of the weighted norm we introduced in (5.28), and the right one is bounded by (thanks to A.6) :

$$\left(\tan(\alpha) \left\| \frac{\partial\Phi_0}{\partial\tilde{Y}} \right\|_{\mathrm{L}^2(\widehat{\mathbf{J}})} + \sqrt{\tan(\alpha)} \int_{\widehat{\Sigma}_+} \Phi_0 T_+ \Phi_0 \right) \|\Phi_\alpha - \Phi_0\|_{\mathrm{H}_\alpha(\widehat{\mathbf{J}})} \tag{5.29}$$

We get then $\|\Phi_\alpha - \Phi_0\|_{\mathrm{H}_\alpha(\widehat{\mathbf{J}})}$ tends to zero, when α tends to zero. That implies $\left\| \dfrac{\partial(\Phi_\alpha - \Phi_0)}{\partial\tilde{X}} \right\|_{\mathrm{L}^2(\widehat{\mathbf{J}})}$ tends to zero. To conclude, we use the following lemma

Lemma 5.5. *for all* $U \in \mathrm{H}^1_0(\widehat{\mathbf{J}})$,

$$\|U\|_{\mathrm{L}^2(\widehat{\Sigma}_+)} \leqslant 2 \left\| \frac{\partial U}{\partial\tilde{X}} \right\|_{\mathrm{L}^2(\widehat{\mathbf{J}})} \tag{5.30}$$

That implies $\|\Phi_\alpha - \Phi_0\|_{\mathrm{L}^2(\widehat{\Sigma}_+)}$ tends to zero, and we get that $\dfrac{K(\alpha)}{\tan(\alpha)} - 1$ tends to zero, when α tends to zero, by using the Cauchy - Schwartz inegality. ■

Proof of the lemma 5.5 : we start from the square norm of the trace of U over $\widehat{\Sigma}_+$, and we apply the mean-value inegality (remember that U vanishes on $\widehat{\Sigma}_0$) :

$$\int_{\widehat{\Sigma}_+} |U|^2 \leqslant \int_{-1}^{0} \int_{y}^{0} 2 \left| \frac{\partial U}{\partial \tilde{X}}(x,y) \right| |U(x,y)| \, dxdy \tag{5.31}$$

and we use the Cauchy - Schwartz on (5.31) to have

$$\|U\|^2_{\mathrm{L}^2(\widehat{\Sigma}_+)} \leqslant 2 \left\| \frac{\partial U}{\partial \tilde{X}} \right\|_{\mathrm{L}^2(\widehat{\mathbf{J}})} \|U\|_{\mathrm{L}^2(\widehat{\mathbf{J}})} \tag{5.32}$$

Then, another application of the mean-value inegality gives

$$\begin{aligned} \int_{\widehat{\mathbf{J}}} |U|^2 &\leqslant \int_{-1}^{0} \int_{y}^{0} 2 \left(\int_{y}^{x} \left| \frac{\partial U}{\partial \tilde{X}}(z,y) \right| |U(z,y)| \, dz \right) dxdy \\ &\leqslant \int_{-1}^{0} \int_{y}^{0} 2 \left(\int_{y}^{0} \left| \frac{\partial U}{\partial \tilde{X}}(z,y) \right| |U(z,y)| \, dz \right) dxdy \end{aligned} \tag{5.33}$$

We can see that the integral over z in (5.33) does not depend on x, so the intergral over x can be bounded by 1. Then, using the Cauchy - Schwartz inegality, (5.33) becomes

$$\|U\|^2_{\mathrm{L}^2(\widehat{\mathbf{J}})} \leqslant 2 \left\| \frac{\partial U}{\partial \tilde{X}} \right\|_{\mathrm{L}^2(\widehat{\mathbf{J}})} \|U\|_{\mathrm{L}^2(\widehat{\mathbf{J}})} \tag{5.34}$$

Then, (5.32) combined with (5.34) gives (5.30), and the proof of the lemma is complete. ■

Remark 5.6. We proved in the proposition 5.4 that the L^2 norm of the derivate of $\dfrac{\partial(\Phi_\alpha - \Phi_0)}{\partial \tilde{X}}$ tends to zero, when α tends to zero. We can ask ourself for which $s \in \mathbb{R}$ the norm $\|\Phi_\alpha - \Phi_0\|_{\mathrm{H}^s(\widehat{\mathbf{J}})}$ tends to zero, when α tends to zero. One remark we can do is that

$$\left\| \frac{\partial \Phi_\alpha}{\partial \tilde{Y}} - \frac{\partial \Phi_0}{\partial \tilde{Y}} \right\|_{\mathrm{L}^2(\widehat{\Sigma}_Y)} = 1 \tag{5.35}$$

By using the trace theorem, one can say that for $s \geqslant 3/2$, the norm $\|\Phi_\alpha - \Phi_0\|_{\mathrm{H}^s(\widehat{\mathbf{J}})}$ does not tend to zero, when α tends to zero. In fact, one over has that

$$\frac{\partial \Phi_\alpha}{\partial \tilde{Y}} - \frac{\partial \Phi_0}{\partial \tilde{Y}} = 1 \quad \text{in } \mathrm{H}^{\frac{1}{2}}(\widehat{\Sigma}_Y) \tag{5.36}$$

This means that the H^1 norm of $\Phi_\alpha - \Phi_0$ does not tend to zero, when α tends to 0. In the proof of the proposition 5.4, we can see that the derivate $\Phi_\alpha - \Phi_0$ over $\tilde{X}$ tends to 0 in L^2 norm , when α tends to 0. This means that the derivate over $\tilde{Y}$ does not tend to 0 in L^2 norm.

Proposition 5.7. *When $\alpha \to \frac{\pi}{2}$, the function Φ_α converges (at least in a L^2 sense) to the function $\Phi_{\frac{\pi}{2}} : (\tilde{X}, \tilde{Y}) \mapsto 0$, and we have $K(\alpha) = o(\tan(\alpha))$*

Proof. The proof is very similar to the proof of the proposition 5.4. One starts from (5.20), divided by $\tan^2(\alpha)$. Formally, if we let α tends to $\frac{\pi}{2}$, we can see that the limit problem obtained is : find $\Phi_{\frac{\Pi}{2}} \in \mathrm{H}_0^1(\widehat{\mathbf{J}})$ such that for all $\mathcal{V} \in \mathrm{H}_0^1(\widehat{\mathbf{J}})$,

$$\int_{\widehat{\mathbf{J}}} \frac{\partial \Phi_{\frac{\pi}{2}}}{\partial \tilde{X}} \frac{\partial \mathcal{V}}{\partial \tilde{X}} = 0 \tag{5.37}$$

Here, it is clear that $\Phi_{\frac{\pi}{2}} \equiv 0$ on $\widehat{\mathbf{J}}$, and the trace of $\Phi_{\frac{\pi}{2}}$ over $\widehat{\Sigma}_+$ is equal to 0.
To prove that Φ_α converges to $\Phi_{\frac{\pi}{2}}$, when α tends to $\frac{\pi}{2}$, we make the difference between the two variationnal formulations (5.20)/ $\tan^2(\alpha)$ and (5.37), to get

$$\begin{aligned} \int_{\widehat{\mathbf{J}}} \Big(\frac{1}{\tan^2(\alpha)} \frac{\partial(\Phi_\alpha - \Phi_{\frac{\pi}{2}})}{\partial \tilde{X}} \frac{\partial \mathcal{V}}{\partial \tilde{X}} + \frac{\partial(\Phi_\alpha - \Phi_{\frac{\pi}{2}})}{\partial \tilde{Y}} \frac{\partial \mathcal{V}}{\partial \tilde{Y}} \Big) \\ + \frac{1}{\tan(\alpha)} \int_{\widehat{\Sigma}_+} \mathcal{V} T_+ (\Phi_\alpha - \Phi_{\frac{\pi}{2}}) = \frac{1}{\tan^2(\alpha)} \int_{\widehat{\Sigma}_+} \mathcal{V} \end{aligned} \tag{5.38}$$

We look here for the problem (5.38) with the weighted norm

$$\|U\|^2_{\mathrm{H}'_\alpha(\widehat{\mathbf{J}})} = \frac{1}{\tan^2(\alpha)} \left\| \frac{\partial U}{\partial \tilde{X}} \right\|^2_{\mathrm{L}^2(\widehat{\mathbf{J}})} + \left\| \frac{\partial U}{\partial \tilde{Y}} \right\|^2_{\mathrm{L}^2(\widehat{\mathbf{J}})} + \frac{1}{\tan(\alpha)} \int_{\widehat{\Sigma}_+} U T_+ U \tag{5.39}$$

After taking $\mathcal{V} = \Phi_\alpha - \Phi_{\frac{\pi}{2}}$ in (5.38), we can see that the left member is the square of the weighted norm defined in (5.39), and the right one can be bounded by $2\|\Phi_\alpha - \Phi_{\frac{\pi}{2}}\|_{\mathrm{H}'_\alpha(\widehat{\mathbf{J}})} / \tan(\alpha)$ (by using again the lemma 5.5). Then one can see that the norm $\|\Phi_\alpha - \Phi_{\frac{\pi}{2}}\|_{\mathrm{H}'_\alpha(\widehat{\mathbf{J}})}$ is uniformly bounded by $\dfrac{2}{\tan(\alpha)}$. This proves then that $\left\| \dfrac{\partial(\Psi_\alpha - \Psi_{\frac{\pi}{2}})}{\partial \tilde{Y}} \right\|_{\mathrm{L}^2(\widehat{\mathbf{J}})}$ tends to zero, when α tends to $\frac{\pi}{2}$, and, by using a result similar to the result of the lemma (5.5), one can see that $\dfrac{K(\alpha)}{\tan(\alpha)}$ tends to zero, when α tends to $\frac{\pi}{2}$. ∎

The results of the propositions 5.3, 5.4 and 5.7 can be illustrated by the figure 5.6

5.2 Study of the reflexion and transmission for small ε

For small values of ε, the transmission and reflexion coefficients are given respectively by (5.5) and (5.6). We neglect here the $O((\varepsilon\omega)^2)$ term. The modulus of the coefficient R is

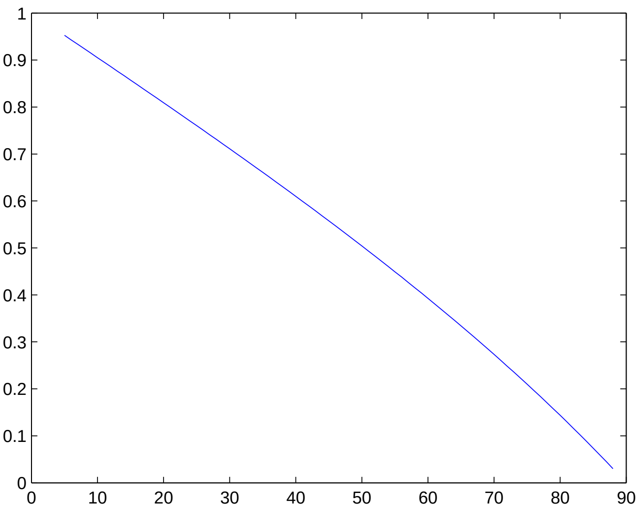

Figure 5.6: Plot of $K(\alpha)/\tan(\alpha)$ with respect to α in degrees

given by $|R_\varepsilon(\alpha,\omega)| = \varepsilon\omega|\tan(\alpha) - K(\alpha)|$. One can see that

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\left(\tan(\alpha) - K(\alpha)\right) &= \frac{\partial}{\partial\alpha}\left(\tan(\alpha)\left(1 - \frac{K(\alpha)}{\tan(\alpha)}\right)\right) \\
&= \left(1+\tan^2(\alpha)\right)\left(1 - \frac{K(\alpha)}{\tan(\alpha)}\right) - \tan(\alpha)\frac{\partial}{\partial\alpha}\left(\frac{K(\alpha)}{\tan(\alpha)}\right) \\
&\geqslant 0
\end{aligned}$$

thanks to the propositions 5.3 and 5.4. Moreover, since $K(0) = 0$, we can say that the function $\alpha \mapsto \tan(\alpha) - K(\alpha)$ is a positive increasing function, and $|R_\varepsilon(\alpha,\omega)| = \varepsilon\omega(\tan(\alpha) - K(\alpha))$. We can see that the reflexion coefficient is an increasing function of α.

6 Numerical computations

6.1 Survey of the exact solution (time harmonic case)

There is two different manners to compute the solution : the first one is to compute the solution of an isotropic equation on a mesh which depends on ε, the second one (which will be used) is to compute the solution of an anisotropic equation (with anisotropic coefficients which depend on ε) on a fixed mesh[1]. We took $\alpha = \pi/4$, $L = 4$, $L' = 5$ for the geometric parameters, and $\omega = 2\pi$ for the wave pulsation. The solutions were computed using the code MONTJOIE[2]

For $\varepsilon = 0.1$, we can see that the solution has a 2D behaviour on the elbow, and has a 1D behaviour on the slots (see the figures 6.1 and 6.2). For $\varepsilon = 0.001$, the variations of the functions in the elbow are so small that we cannot see them (see the figures 6.3 and 6.4).

Remark 6.1. We cannot take ε smaller than 10^{-6}, because of the anisotropic coefficients : when we have $\alpha \neq \pi/4$, we have numerical errors that give a numerical solution which does not be the exact solution.

6.2 Survey of the exact solution (time domain case)

In this section, we go back to the time domain (t will denote the time) and the wave equation on which it is easier to illustrate out results. The approximate 1D problem we consider is

$$\left\{\begin{array}{ll} \dfrac{\partial^2 \tilde{u}^\varepsilon}{\partial t^2} - \dfrac{\partial^2 \tilde{u}^\varepsilon}{\partial s^2} = 0, & \text{in } \mathbb{R}_+ \times \; (]-L^-, 0\,[\,\cup\,]\,0, L^+[)\ , \\[2ex] [\tilde{u}^\varepsilon] - \varepsilon\, K(\alpha) \left\langle \dfrac{\partial \tilde{u}^\varepsilon}{\partial s} \right\rangle = 0, \quad \left[\dfrac{\partial \tilde{u}^\varepsilon}{\partial s}\right] - \varepsilon \tan\alpha \left\langle \dfrac{\partial^2 \tilde{u}^\varepsilon}{\partial t^2} \right\rangle = 0, & \text{at } s = 0. \end{array}\right. \tag{6.1}$$

For the source term, we consider initial conditions chosen in such a way that for initial values of t the solution corresponds to an incident wave in $s < 0$

$$\tilde{u}^\varepsilon(s,t) = u_0(s-t) \quad \Big(\quad \Longleftrightarrow \quad \tilde{u}^\varepsilon(s,0) = u_0(s), \quad \frac{\partial \tilde{u}^\varepsilon}{\partial t}(s,0) = -u_0'(s) \ \Big)$$

where u_0 is a smooth "Gaussian like" function compactly supported in $s < 0$. For the exact or reference solution, we consider the wave equation in the domain Ω^ε with $L_\pm = +\infty$ and Neumann boundary conditions and with the "same initial conditions" (we mean here that the initial data are independent of the transverse coordinate in the slot).

[1] This mesh, of course, depends on the value of α
[2] homepage : *http://www-rocq.inria.fr/poems/montjoie/*

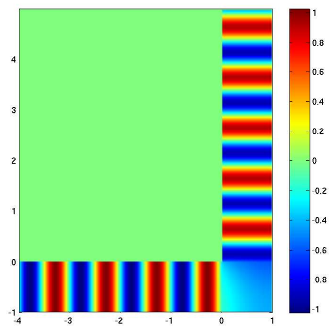

Figure 6.1: Plot of the real part of the solution with $\varepsilon = 0.1$

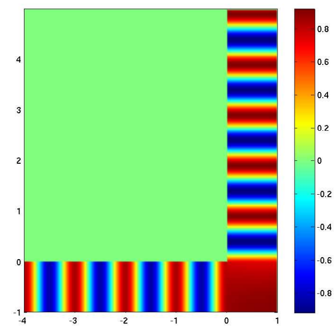

Figure 6.2: Plot of the imaginary part of the solution with $\varepsilon = 0.1$

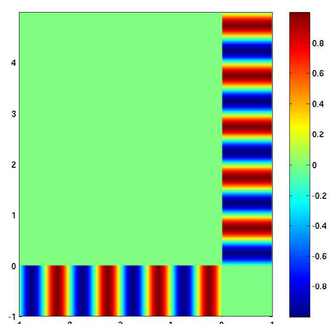

Figure 6.3: Plot of the real part of the solution with $\varepsilon = 0.001$

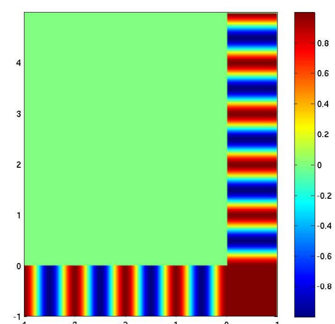

Figure 6.4: Plot of the imaginary part of the solution with $\varepsilon = 0.001$

We used the code MONTJOIE to compute the 2D exact solution. We plot on figures 6.5 to 6.7 the solution at a time t large enough to see the effects of the junction. We take $\varepsilon = \lambda/10$ (where λ is the wave length of the Cauchy data).

Figures 6.5 to 6.7 : 2D Computations of the exact solution ($\varepsilon = 0.1\lambda$).

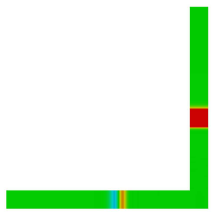

Figure 6.5: $\alpha = \frac{\pi}{4}$

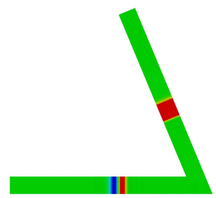

Figure 6.6: $\alpha = \frac{5\pi}{16}$

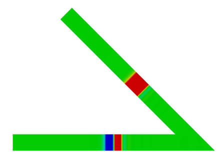

Figure 6.7: $\alpha = \frac{3\pi}{8}$

On the figures 6.5 to 6.7, we plot the solution on the scaled set $\widehat{\Omega}_- \cup \widehat{J}_\alpha \cup \widehat{\Omega}_-$, for differents values of α, and we use the same range to see the effect of the junction on the reflection (the green color corresponds to the value 0). We can see about the reflection wave that :

- The form is the derivate of the "Gaussian like" function
- Its amplitude increases with the value of α

6.3 Numerical results on the improved 1D model

For the 1D model (6.1), we saw in the section 5 that, for small values of $\alpha\varepsilon/\lambda$, the reflection phenomen corresponds in first approximation to a derivation with respect to time of the incident signal. To see this phenomena, we compute the function $\tilde{u}^\varepsilon$ solution of (6.1) with the initial data $u_0(s) = \exp(-5(s + L/2)^2)$, and we plot $\varepsilon^{-1}\tilde{u}^\varepsilon(t, s)$ as function of t, with $s = -3L/4$, for differents values of ε, and we compared with the translation of $-(\tan(\alpha) - K(\alpha))\, u'(s)$

We can see in figures 6.8 and 6.9 that our comments about the reflection phenomena are numerically discernible.

6.4 Convergence between the approximated model and the exact model

We wish to check numerically the error (1.20) (theorem 1.5). We take $\delta = 0.2$, $\alpha = \frac{\pi}{4}$, and proceed as follow :

(i) We compute $K(\alpha)$ once for all with a FreeFem++[3] script.

[3] url : *http://www.freefem.org/ff++/*

Figures 6.8 and 6.9 : plot of $\varepsilon^{-1}\tilde{u}^{\varepsilon}(t,s)$ as function of t, with $s=-\frac{3L}{4}$

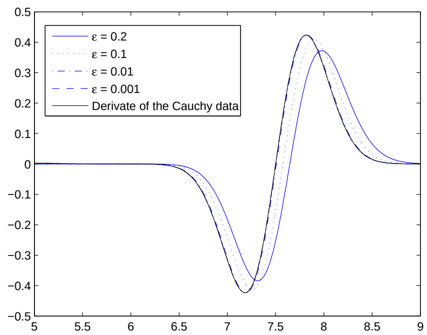

Figure 6.8: $\alpha=\frac{\pi}{4}$

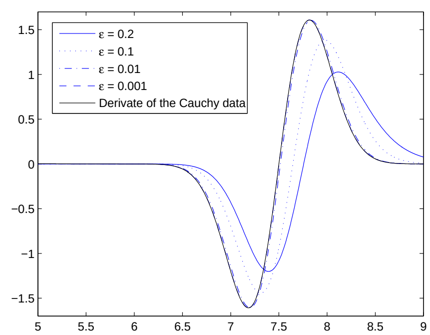

Figure 6.9: $\alpha=\frac{3\pi}{8}$

(ii) For differents values of $\varepsilon\in\,]10^{-3},10^{-1}[$, we compute u^{ε} solution of (1.3) and $\tilde{u}^{\varepsilon}$ solution of (1.16, 1.17), and we build $\widehat{u}_{\pm}^{\varepsilon,app}$ by (1.18)

(iii) We compute $\displaystyle\sum_{\pm}\|\widehat{u}_{\pm}^{\varepsilon,app}-\widehat{u}_{\pm}^{\varepsilon}\|_{\mathrm{H}^1(\widehat{\Omega}_{\pm}^{\delta})}$

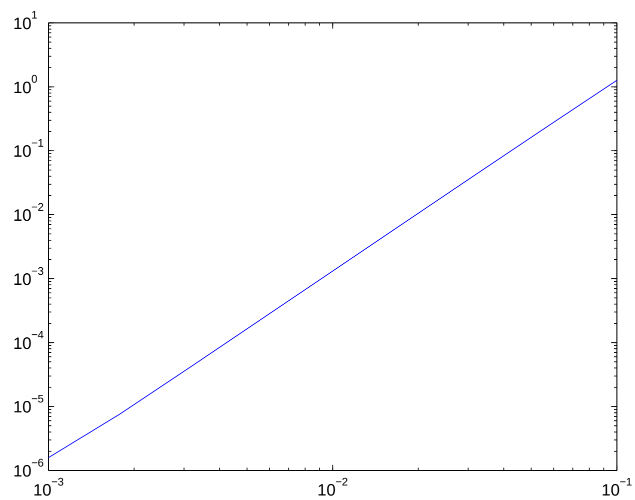

Figure 6.10: $\displaystyle\sum_{\pm}\|\widehat{u}_{\pm}^{\varepsilon,app}-\widehat{u}_{\pm}^{\varepsilon}\|_{\mathrm{H}^1(\widehat{\Omega}_{\pm}^{\delta})}$ as function of ε (logscale)

We can see on figure 6.10 that the error $\sum_{\pm} \|\widehat{u}_{\pm}^{\varepsilon,app} - \widehat{u}_{\pm}^{\varepsilon}\|_{\mathrm{H}^1(\widehat{\Omega}_{\pm}^{\delta})}$ behaves as $C\varepsilon^{\beta}$, with $\beta \simeq 3$, as expected.

A Some properties about the non-local DtN operator

A.1 Definition of the functionnal framework and the DtN operator

It is well known that a function in $\mathrm{L}^2(]-1,0[)$ can be decomposed in form of cosines and sines. For our interesting case, we denote $\mathrm{L}^2_{\cos}(]-1,0[)$ the following set

$$\mathrm{L}^2_{\cos}(]-1,0[) = \left\{ U \in \mathrm{L}^2(]-1,0[) \text{ such as } \forall p \in \mathbb{N}^*,\ \int_{-1}^0 U(y)\sin(p\pi y)dy = 0 \right\} \tag{A.1}$$

In the same way, we can define, for $s > 0$, the following set

$$\mathrm{H}^s_{\cos}(]-1,0[) = \left\{ U \in \mathrm{H}^s(]-1,0[) \text{ such as } \forall p \in \mathbb{N}^*,\ \int_{-1}^0 U(y)\sin(p\pi y)dy = 0 \right\} \tag{A.2}$$

Note that, for $s \leqslant s'$, $\mathrm{H}^{s'}_{\cos}(]-1,0[) \subset \mathrm{H}^s_{\cos}(]-1,0[)$. Let us now introduce the orthonormal basis of $\mathrm{L}^2_{\cos}(]-1,0[)$ given by

$$w_0(\bullet) = 1, \quad w_p(\bullet) = \sqrt{2}\cos(p\pi\bullet),\ p = 1,2,3\ldots \tag{A.3}$$

With the basis (A.3), a natural norm on the space $\mathrm{H}^s_{\cos}(]-1,0[)$ is

$$\|\varphi\|^2_{\mathrm{H}^s_{\cos}(]-1,0[)} = \sum_{p\in\mathbb{N}} (1+p^2)^s \left|(\varphi, w_p)_0\right|^2 \tag{A.4}$$

where

$$(\varphi, w_p)_0 = \int_{-1}^0 \varphi(y) w_p(y) dy \tag{A.5}$$

We can now define the following DtN operator :

Definition A.1. For any function φ in $\mathrm{H}^s_{\cos}(]-1,0[)$, we can define the function $T\varphi$ as

$$\varphi \Longmapsto T\varphi = \sum_{p\in\mathbb{N}} p\pi (\varphi, w_p)_0\ w_p \tag{A.6}$$

Remark A.2. We cas easily see that, at least in a formally sense, $T\phi = 0$ for $\phi \mathrm{H}^s_{\cos}(]-1,0[)$ if and only if ϕ is constant.

A.2 Properties about the DtN operator

Here, we will express and prove some properties about the DtN operator T.

Proposition A.3. *For any* $s \geqslant 0$, $T \in \mathcal{L}(\mathrm{H}^s_{\cos}(]-1,0[),\ \mathrm{H}^{s-1}(]-1,0[))$. *Moreover, for any* $s \geqslant 1$, $T \in \mathcal{L}(\mathrm{H}^s_{\cos}(]-1,0[),\ \mathrm{H}^{s-1}_{\cos}(]-1,0[))$.

Proof. We treat differently the case $s \geqslant 1$ and $s < 1$.

- The case $s \geqslant 1$: let $\varphi \in (\mathrm{H}^s_{\cos}(]-1,0[)$ and let us show that

$$\|T\varphi\|_{\mathrm{H}^{s-1}_{\cos}(]-1,0[)} \leqslant \pi \|\varphi\|_{\mathrm{H}^{s}_{\cos}(]-1,0[)} \tag{A.7}$$

We initially start from

$$\begin{aligned}\|T\varphi\|^2_{\mathrm{H}^{s-1}_{\cos}(]-1,0[)} &= \sum_{p\in\mathbb{N}} (1+p^2)^{s-1} \left|(T\varphi, w_p)_0\right|^2 \\ &= \sum_{p\in\mathbb{N}} (1+p^2)^{s-1} \left|\sum_{q\in\mathbb{N}} q\pi(\varphi, w_q)_0 (w_q, w_p)_0\right|^2\end{aligned}$$

Since $(w_p)_{p\in\mathbb{N}}$ is an orthonormal basis on $\mathrm{L}^2_{\cos}(]-1,0[)$, we get that

$$\|T\varphi\|^2_{\mathrm{H}^{s-1}_{\cos}(]-1,0[)} = \sum_{p\in\mathbb{N}} \pi(1+p^2)^{s-1} p^2 \left|(\varphi, w_p)_0\right|^2$$

and to raise up p^2 by $1+p^2$ to get (A.7)

- The case $s \leqslant 1$: let $\varphi \in (\mathrm{H}^s_{\cos}(]-1,0[)$ and let us show that, that for all $\psi \in (\mathrm{H}^{1-s}_{\cos}(]-1,0[)$,

$$\left|\int_{-1}^{0} \psi(y) T\varphi(y) dy\right| \leqslant \pi \|\varphi\|_{\mathrm{H}^{s}_{\cos}(]-1,0[)} \|\psi\|_{\mathrm{H}^{s}_{\cos}(]-1,0[)} \tag{A.8}$$

We initially from

$$\begin{aligned}\left|\int_{-1}^{0} \psi(y) T\varphi(y) dy\right| &= \left|\sum_{p\in\mathbb{N}} p\pi(\varphi, w_p)_0 (\psi, w_p)_0\right| \\ &\leqslant \pi \sum_{p\in\mathbb{N}} \sqrt{1+p^2} \left|(\varphi, w_p)_0\right| \left|(\psi, w_p)_0\right| \\ &\leqslant \pi \sum_{p\in\mathbb{N}} (\sqrt{1+p^2})^s \left|(\varphi, w_p)_0\right| (\sqrt{1+p^2})^{1-s} \left|(\psi, w_p)_0\right|\end{aligned}$$

We use then the Cauchy - Schwartz inegality and the definition of the $\mathrm{H}^s_{\cos}$ norms to get (A.8).

■

Remark A.4. Of course, the interesting cases (that we exploit) are for $s = \frac{1}{2}$ and $s = 1$.

Proposition A.5. *For all* $\varphi \in \mathrm{H}^{\frac{1}{2}}_{\cos}(]-1,0[)$, *we have*

$$\int_{-1}^{0} \overline{\varphi}(y) T\varphi(y) dy \geqslant 0 \tag{A.9}$$

Proof. Let $\varphi \in \mathrm{H}^{\frac{1}{2}}_{\cos}(]-1,0[)$ given. Thanks to the proposition A.3 with $s = \frac{1}{2}$, the quantity $\displaystyle\int_{-1}^{0} \overline{\varphi}(y) T\varphi(y)dy$ is finite. Then we have

$$\int_{-1}^{0} \overline{\varphi}(y) T\varphi(y)dy = \pi \sum_{p\in\mathbb{N}} p(\overline{\varphi}, w_p)_0 (\varphi, w_p)_0 = \pi \sum_{p\in\mathbb{N}} p\overline{(\varphi, w_p)_0}(\varphi, w_p)_0 \tag{A.10}$$

and (A.9) is proved. ∎

Proposition A.6. *Let φ and ψ in $\mathrm{H}^{\frac{1}{2}}_{\cos}(]-1,0[)$, then*

$$\left|\int_{-1}^{0} \psi(y) T\varphi(y)dy\right|^2 \leqslant \int_{-1}^{0} \overline{\varphi}(y) T\varphi(y)dy \int_{-1}^{0} \overline{\psi}(y) T\psi(y)dy \tag{A.11}$$

Proof. One starts from

$$\left|\int_{-1}^{0} \psi(y) T\varphi(y)dy\right|^2 = \left|\pi \sum_{p\in\mathbb{N}} p(\varphi, w_p)_0 (\psi, w_p)_0\right|^2 \tag{A.12}$$

By using the Cauchy - Schwartz inegality, (A.12) becomes

$$\left|\int_{-1}^{0} \psi(y) T\varphi(y)dy\right|^2 \leqslant \pi^2 \sum_{p\in\mathbb{N}} p\left|(\varphi, w_p)_0\right|^2 \sum_{p\in\mathbb{N}} p\left|(\psi, w_p)_0\right|^2 \tag{A.13}$$

We use then (A.10) to conclude. ∎

B The stability result

For the initial problem (1.3), it is possible to introduce its variationnal formulation : find $u^\varepsilon \in \mathrm{H}^1\left(\Omega_\alpha^\varepsilon;\mathbb{C}\right)$ such that for all $v \in \mathrm{H}^1\left(\Omega_\alpha^\varepsilon;\mathbb{C}\right)$,

$$\frac{1}{\varepsilon}\left(\int_{\Omega_\alpha^\varepsilon}\left(\nabla u^\varepsilon \nabla \overline{v} - \omega^2 u^\varepsilon \overline{v}\right) - \int_{\Gamma_+^\varepsilon} \imath\omega u^\varepsilon \overline{v}\right) = \frac{1}{\varepsilon}\int_{\Gamma_-^\varepsilon} f\overline{v} \tag{B.1}$$

We introduced in (B.1) the factor ε^{-1} to have the right member independant of ε, when we take $\overline{v} = 1$ Let then denote by $a^\varepsilon(u^\varepsilon, v)$ the left member of (B.1). It is also natural to introduce the space

$$\mathrm{H}^1_{\lim}(\widehat{\Omega}_\alpha) = \left\{u \in \mathrm{H}^1(\widehat{\Omega}_\alpha) \text{ such as } u(s,\widehat{\nu}) = u(s) \text{ in } \widehat{\Omega}_\pm \text{ and } u(X,Y) = u(0) \text{ in } \widehat{\mathcal{J}}_\alpha\right\} \tag{B.2}$$

Thanks to trace theorem, for a given $\alpha \in]-\frac{\pi}{2},\frac{\pi}{2}[$, an equivalent norm for the H^1 norm for functions un $\mathrm{H}^1_{\lim}(\widehat{\Omega}_\alpha)$ is the $\mathrm{H}^1(S_\pm)$ norm. Let us write a^ε as the left member of (B.1). One can see that, for any given ε, the Hilbert spaces $\mathrm{H}^1\left(\Omega_\alpha^\varepsilon;\mathbb{C}\right)$ and $\mathrm{H}^1\left(\widehat{\Omega}_\alpha;\mathbb{C}\right)$ (when $\widehat{\Omega}_\alpha$ is the scaled set, i.e. by taking $\varepsilon = 1$) are equivalent vectorial spaces. By using the Riesz representation theorem, forall $u \in \mathrm{H}^1\left(\widehat{\Omega}_\alpha;\mathbb{C}\right)$, there exists a unique $A^\varepsilon u \in \mathrm{H}^1\left(\widehat{\Omega}_\alpha;\mathbb{C}\right)$ such that, for all $v \in \mathrm{H}^1\left(\widehat{\Omega}_\alpha;\mathbb{C}\right)$,

$$a^\varepsilon(u,v) = \langle A^\varepsilon u, v\rangle_{\mathrm{H}^1\left(\widehat{\Omega}_\alpha;\mathbb{C}\right)} \tag{B.3}$$

The stability result can be anounced as below :

Proposition B.1. *There exists $C > 0$ independant of ε such as $\|A^\varepsilon\|^{-1}_{\mathrm{H}^1\left(\widehat{\Omega}_\alpha;\mathbb{C}\right)} \leqslant C$*

Proof. By contradiction, if the proposition B.1 is not true, there exists a sequence v^ε such that

$$\|v^\varepsilon\|_{\mathrm{H}^1\left(\widehat{\Omega}_\alpha;\mathbb{C}\right)} = 1 \quad \text{and} \quad \|A^\varepsilon v^\varepsilon\|_{\mathrm{H}^1\left(\widehat{\Omega}_\alpha;\mathbb{C}\right)} \to 0 \text{ when } \varepsilon \to 0 \tag{B.4}$$

Even if we have to extract a subsequence by using some compacity, we can suppose that v^ε converges to a limit function v^0, weakly in $\mathrm{H}^1\left(\widehat{\Omega}_\alpha;\mathbb{C}\right)$ and strongly in $\mathrm{L}^2\left(\widehat{\Omega}_\alpha;\mathbb{C}\right)$. Then by using the Reisz representation by taking $u = v = v^\varepsilon$, we get that

$$\begin{aligned} &\left\|\frac{\partial v^\varepsilon}{\partial s}\right\|^2_{\mathrm{L}^2\left(\widehat{\Omega}_\pm;\mathbb{C}\right)} + \frac{1}{\varepsilon^2}\left\|\frac{\partial v^\varepsilon}{\partial \widehat{\nu}}\right\|^2_{\mathrm{L}^2\left(\widehat{\Omega}_\pm;\mathbb{C}\right)} + \frac{1}{\varepsilon}\|\nabla v^\varepsilon\|^2_{\mathrm{L}^2\left(\widehat{\mathcal{J}}_\alpha;\mathbb{C}\right)} \\ = \quad &\omega^2\|v^\varepsilon\|^2_{\mathrm{L}^2\left(\widehat{\Omega}_\pm;\mathbb{C}\right)} + \varepsilon\omega^2\|v^\varepsilon\|^2_{\mathrm{L}^2\left(\widehat{\mathcal{J}}_\alpha;\mathbb{C}\right)} + \imath\omega\int_{\widehat{\Gamma}_+}|v^\varepsilon|^2 + \langle A^\varepsilon v^\varepsilon, v^\varepsilon\rangle_{\mathrm{H}^1\left(\widehat{\Omega}_\alpha;\mathbb{C}\right)} \end{aligned} \tag{B.5}$$

The right member of (B.5) is bounded by $\left(\omega^2 + \varepsilon\omega + \omega\right)\|v^\varepsilon\|^2_{\mathrm{H}^1\left(\widehat{\Omega}_\alpha;\mathbb{C}\right)} + \|A^\varepsilon v^\varepsilon\|_{\mathrm{H}^1\left(\widehat{\Omega}_\alpha;\mathbb{C}\right)}\|v^\varepsilon\|_{\mathrm{H}^1\left(\widehat{\Omega}_\alpha;\mathbb{C}\right)}$, which can be uniformy bounded (in fact, we have the hypothesis (B.4) which gives that

$\|A^\varepsilon v^\varepsilon\|_{\mathrm{H}^1(\widehat{\Omega}_\alpha;\mathbb{C})}$ is lesser than 1, for ε small enough). Then the left member of (B.5) is also uniformly bounded. The weak convergence of v^ε to v^0 gives immediately that

$$\left\|\frac{\partial v^0}{\partial \widehat{\nu}}\right\|_{\mathrm{L}^2(\widehat{\Omega}_\pm;\mathbb{C})} = \left\|\nabla v^0\right\|_{\mathrm{L}^2(\widehat{\mathcal{J}}_\alpha;\mathbb{C})} = 0 \tag{B.6}$$

and we even have

$$\lim_{\varepsilon\to 0}\left\|\frac{\partial v^\varepsilon}{\partial \widehat{\nu}}\right\|_{\mathrm{L}^2(\widehat{\Omega}_\pm;\mathbb{C})} = \lim_{\varepsilon\to 0}\|\nabla v^\varepsilon\|_{\mathrm{L}^2(\widehat{\mathcal{J}}_\alpha;\mathbb{C})} = 0 \tag{B.7}$$

We get then that $v^0 \in \mathrm{H}^1_{\lim}(\widehat{\Omega}_\alpha)$. Once we got this result, we use that the function v^ε weakly converges to v^0 to get

$$\langle v^\varepsilon, w\rangle_{\mathrm{H}^1(\widehat{\Omega}_\alpha;\mathbb{C})} \to \langle v^0, w\rangle_{\mathrm{H}^1(\widehat{\Omega}_\alpha;\mathbb{C})}, \quad \forall v \in \mathrm{H}^1\left(\widehat{\Omega}_\alpha;\mathbb{C}\right) \tag{B.8}$$

and we choose $w = A^{\varepsilon'}\Phi$, with $\Phi \in \mathrm{H}^1(\widehat{\Omega}_\alpha)$. By using the Riesz representation (B.3) and the fact that $\langle u,v\rangle_{\mathrm{H}^1} = \overline{\langle v,u\rangle_{\mathrm{H}^1}}$, we get that

$$\left\langle A^{\varepsilon'}v^\varepsilon, \Phi\right\rangle_{\mathrm{H}^1(\widehat{\Omega}_\alpha;\mathbb{C})} \to \left\langle A^{\varepsilon'}v^0, \Phi\right\rangle_{\mathrm{H}^1(\widehat{\Omega}_\alpha;\mathbb{C})}, \quad \forall\Phi \in \mathrm{H}^1\left(\widehat{\Omega}_\alpha;\mathbb{C}\right) \text{ and } \varepsilon \to 0 \tag{B.9}$$

One can also see that, for all function $\Phi \in \mathrm{H}^1_{\lim}\left(\widehat{\Omega}_\alpha\right)$, the operator $\left\langle A^{\varepsilon'}v^\varepsilon, \Phi\right\rangle_{\mathrm{H}^1(\widehat{\Omega}_\alpha;\mathbb{C})}$ tends to the operator $\left\langle A^0 v^\varepsilon, \Phi\right\rangle_{\mathrm{H}^1(\widehat{\Omega}_\alpha;\mathbb{C})}$ as ε' tends to zero (this is natural to consider Φ in a such space, since v^0 belongs to this space). By using some diagonal extraction, and the fact that $\|A^\varepsilon v^\varepsilon\|_{\mathrm{H}^1(\widehat{\Omega}_\alpha;\mathbb{C})} \to 0$, we get that $a^0(v^0,\Phi) = 0$ for every test function $\Phi \in \mathrm{H}^1_{\lim}\left(\widehat{\Omega}_\alpha\right)$. This gives immediately that $v_0 = 0$ (we can compute here the solution explicitely), then we have

$$\lim_{\varepsilon\to 0}\|v^\varepsilon\|_{\mathrm{L}^2(\widehat{\Omega}_\alpha;\mathbb{C})} = 0 \tag{B.10}$$

To finish, we get the real part of (B.5), and we can use some trivial inegalities to get

$$\left\|\frac{\partial v^\varepsilon}{\partial s}\right\|^2_{\mathrm{L}^2(\widehat{\Omega}_\pm;\mathbb{C})} \leqslant \omega^2(1+\varepsilon)\,\|v^\varepsilon\|^2_{\mathrm{L}^2(\widehat{\Omega}_\alpha;\mathbb{C})} + \|A^\varepsilon v^\varepsilon\|_{\mathrm{H}^1(\widehat{\Omega}_\alpha;\mathbb{C})}\,\|v^\varepsilon\|_{\mathrm{H}^1(\widehat{\Omega}_\alpha;\mathbb{C})} \tag{B.11}$$

By using (B.10) and (B.4), we have that

$$\lim_{\varepsilon\to 0}\left\|\frac{\partial v^\varepsilon}{\partial s}\right\|_{\mathrm{L}^2(\widehat{\Omega}_\pm;\mathbb{C})} = 0 \tag{B.12}$$

Finally, by using (B.10), (B.12) and (B.7), we give that $\|v^\varepsilon\|_{\mathrm{H}^1(\widehat{\Omega}_\alpha;\mathbb{C})} \to 0$, then it is in contradiction with the fact that $\|v^\varepsilon\|_{\mathrm{H}^1(\widehat{\Omega}_\alpha;\mathbb{C})} = 1$. ■

C Convergence of the 1D approximate model

Here we will prove the theorem 4.1 In the sequel, we will denote $\mathcal{H}$ the space $\mathrm{H}^1(S_-) \times \mathrm{H}^1(S_+)$, and we put the following norm

$$\|u\|^2_{\mathcal{H},\varepsilon} = \|u\|^2_{\mathrm{H}^1(S_-)} + \|u\|^2_{\mathrm{H}^1(S_+)} + \frac{1}{\varepsilon K(\alpha)} |[u]|^2 \tag{C.1}$$

Note that, for a given ε, this norm is strictly equivalent to the classical norm $\mathrm{H}^1(S_\pm)$, and moreover we have

$$\|u\|_{\mathrm{H}^1(S_\pm)} \leqslant \|u\|_{\mathcal{H},\varepsilon} \tag{C.2}$$

The other inegality degenerates when ε tends to zero.

The problem (1.16, 1.17) into its variationnal form can be written as : find $\tilde{u}^\varepsilon \in \mathcal{H}$ such that, for all $v \in \mathcal{H}$

$$\int_{S_\pm} \left(\frac{\partial \tilde{u}^\varepsilon}{\partial s} \frac{\partial \overline{v}}{\partial s} - \omega^2 \tilde{u}^\varepsilon \overline{v} \right) + \frac{1}{\varepsilon K(\alpha)} [\tilde{u}^\varepsilon] [\overline{v}] - \varepsilon \omega^2 \tan(\alpha) \langle \tilde{u}^\varepsilon \rangle \langle \overline{v} \rangle - \imath\omega \tilde{u}^\varepsilon(L_+) \overline{v}(L_+) = f \overline{v}(-L_-) \tag{C.3}$$

Let denote $\tilde{a}^\varepsilon(\tilde{u}^\varepsilon, v)$ the left member of (C.3), and $l(v)$ the right one. As for the theorem 3.10, we will prove by using two steps : a stability result (section C.1), and a consistency result (section C.2).

C.1 The stability result for the 1D approximate model

The stability result can be express as the following

Proposition C.1. *There exists a constant independant of ε such that, for all $u \in \mathcal{H}$,*

$$\|u\|_{\mathcal{H},\varepsilon} \leqslant |\tilde{a}^\varepsilon(u, \bullet)|_{\mathcal{H},\varepsilon} \tag{C.4}$$

Proof. By contradition, if the proposition C.1 is false, we can find a sequence $u^\varepsilon \in \mathcal{H}$ such that

$$\|u^\varepsilon\|_{\mathcal{H},\varepsilon} = 1 \quad \text{and} \quad \tilde{a}^\varepsilon(u^\varepsilon, \bullet) \to 0 \tag{C.5}$$

By using the compactness of $\mathrm{H}^1(S_\pm)$ since we have (C.2), there exists a subsequence (that we still denote by u^ε) such that convergerges to u^0 weeakly in $\mathrm{H}^1(S_\pm)$ and strongly in $\mathrm{L}^2(S_\pm)$. Next, using the fact that $\|u^\varepsilon\|_{\mathcal{H},\varepsilon} = 1$ allows us to write that

$$|[u^\varepsilon]| \leqslant \sqrt{\varepsilon K(\alpha)} \tag{C.6}$$

Since the application $[\bullet] \in \mathcal{L}(\mathrm{H}^1(S_\pm), \mathbb{C})$ and thanks to the weakly convergence, $[u^\varepsilon]$ converges weakly to $[u^0]$ in $\mathbb{C}$, and using the inegality (C.6), we get that $[u^0] = 0$. Then, by taking test functions v that satisfies $[v] = 0$ for $\tilde{a}^\varepsilon$, and using the weakly convergence, we get that

$$\int_{S_\pm} \left(\frac{\partial u^0}{\partial s} \frac{\partial \overline{v}}{\partial s} - \omega^2 u^0 \overline{v} \right) - \imath\omega u^0(L_+) \overline{v}(L_+) = 0 \tag{C.7}$$

The problem (C.7) is well posed and admits a unique solution which is $u^0 = 0$. To conclude, we have, by using again the definition of $\tilde{a}^\varepsilon$,

$$\left\| \frac{\partial u^\varepsilon}{\partial s} \right\|^2_{\mathrm{L}^2(S_\pm)} + \frac{1}{\varepsilon K(\alpha)} \left| [u^\varepsilon] \right|^2 = Re(\tilde{a}^\varepsilon(u^\varepsilon, u^\varepsilon)) + \omega^2 \left\| u^\varepsilon \right\|^2 + \varepsilon \omega^2 \tan(\alpha) \left| \langle u^\varepsilon \rangle \right|^2 \tag{C.8}$$

The right member of (C.8) tends to 0 as ε tends to 0 thanks to the hypothesis (C.5) and the strongly convergence of u^ε to 0 for the L^2 norm. Then the left members tends also to 0. We finally get that $\|u^\varepsilon\|_{\mathcal{H},\varepsilon} \to 0$ as $\varepsilon \to 0$, which is in contradiction with (C.5). ■

C.2 The consistency result for the 1D approximate model

Here the error can be get directly. One starts from (after computation)

$$\tilde{a}^\varepsilon \left(\sum_{m=0}^{k} (\varepsilon\omega)^m \tilde{u}^m, v \right) = \tilde{a}^\varepsilon(\tilde{u}^\varepsilon, \overline{v}) - (\varepsilon\omega)^k \left\langle \frac{\partial \tilde{u}^k}{\partial s} \right\rangle [\overline{v}] - \omega \tan(\alpha) (\varepsilon\omega)^{k+1} \langle \tilde{u}^\varepsilon \rangle \langle \overline{v} \rangle \tag{C.9}$$

By using the fact that $| \langle \overline{v} \rangle | \leqslant C \varepsilon^{1/2} \|v\|_{\mathcal{H},\varepsilon}$, and by using trace theorems, we can raise up (C.9) by

$$\left| \tilde{a}^\varepsilon \left(\tilde{u}^\varepsilon - \sum_{m=0}^{k} (\varepsilon\omega)^m \tilde{u}^m, v \right) \right| \leqslant C_k \varepsilon^{k+1/2} \|v\|_{\mathcal{H},\varepsilon} \tag{C.10}$$

By using the stability result given by the proposition C.1 and the inegality (C.2), one gets that

$$\left\| \tilde{u}^\varepsilon - \sum_{m=0}^{k} (\varepsilon\omega)^m \tilde{u}^m \right\|_{\mathrm{H}^1(S_\pm)} \leqslant C_k \varepsilon^{k+1/2} \tag{C.11}$$

The result of the theorem 4.1 is obtained simply by using a triangular inegality, and the fact that $\tilde{u}^{k+1}$ does not depend on ε.

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